Handout for: What would count as Ibn Sīnā having first order logic? (Munich, March 2012

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1 Texts

1.1 Ibn Sīnā on Euclid

Ibn Sīnā *Qiyās* ix.3, 433.5–8 (my translation)

It has been made clear to you that there is no recombinant syllogism with a single premise, nor is there one with more than two premises. But you still have the option of raising a doubt and saying: We have sometimes seen syllogistic discourse in which a proof is devised which has a single goal but more than two premises. There are demonstrations of this kind in the geometrical text *Elements*, and elsewhere.

1.2 Ibn Sīnā on transitivity of equivalence

Ibn Sīnā *Qiyās* i.6 and his student Bahmanyār ibn al-Marzubān on transitivity of equality. The translation of Ibn Sīnā is mine; that of Bahmanyār is slightly adjusted from Khaled el-Rouayheb, *Relational Syllogisms and the History of Arabic Logic, 900–1900*, Brill 2010, p. 24.

(Ibn Sīnā) Thus when you say

$$C \text{ is equal to } B \text{ and } B \text{ is equal to } D, \text{ so } C \text{ is equal to } D. \tag{1}$$

this is complete for you only when you become aware that C is equal to something equal to D, and that a thing that is equal to something equal [to X] is itself equal [to X].

(Bahmanyār) If you say 'C is equal to B and B is equal to D, therefore C is equal to D' you have not made explicit one of the two premises. For the form of the syllogism to be a complete syllogism we say 'C is something that is equal to B which is something that is equal to D, and anything equal to something equal [to X] is equal [to X], and so C is equal to D'. This is as if someone said, 'C is equal to something equal to D, and everything equal to something equal to D is equal to D, so C is equal to D'.

1.3 Ibn Sīnā on compound syllogisms

From Ibn Sīnā *Qiyās* ix.3, on counting the size of a compound syllogism (my translation).

In the case where the other [proximate premise] has to be derived [as well], a syllogism with two premises is introduced in order to derive it. Then at one level there are four premises and two conclusions, and at the second level there are two premises and a single conclusion. So the compound [syllogism] contains six premises altogether and three conclusions altogether. The number of conclusions is half the number of premises. Each of the [simple] syllogisms contains three terms and a conclusion. Suppose in fact that each [proximate] premise [is proved by] a syllogism, and the two [proximate] premises share a term. Then there are six terms, except that one of them is shared in the middle, so there are five terms. The shared term and the term at one end of the five give rise to one proximate premise, and the shared term and the other end term give rise to the other [proximate] premise. The two end terms of the five give rise to the goal which is the target of the compound syllogism.

1.4 Ibn Sīnā on local validation

From Ibn Sīnā *Autobiography*, trans. D. Gutas, *Avicenna and the Aristotelian Tradition*, Brill, Leiden 1988, pp. 27f. Gutas' 'Ascertained' is better translated 'verified', a near-synonym of 'validated'. Also Gutas explains 'conditions' (of the premises) as 'modalities'; but in Ibn Sīnā's nor-

mal usage the word (*šarṭ*) means side-conditions, particularly ones that the speaker intended but didn't say.

8. The next year and a half I devoted myself entirely to reading Philosophy: I read Logic and all the parts of philosophy once again. During this time I did not sleep completely through a single night, or occupy myself with anything else by day. I compiled a set of files for myself, and for each argument that I examined, I recorded the syllogistic premises it contained, the way in which they were composed, and the conclusions which they might yield, and I would also take into account the conditions of its premises until I had Ascertained that particular problem. … So I continued until all the Philosophical Sciences became deeply rooted in me and I understood them as much as is humanly possible. Everything that I knew at that time is just as I know it now; I have added nothing more to it to this day.

From Ibn Sīnā *Qiyās* ix.9, 490.7–10

It can happen that the given syllogism is compound, so that it has to be analysed first into [simple] syllogisms. Then when one [simple] syllogism is found, that syllogism creates a prejudice that the second syllogism is in the same figure. One should disregard that, since we know that [compound] syllogisms can be compounded from [simple syllogisms in] different figures and moods. Rather one has to consider each [simple] syllogism as it stands, apart from any other.

1.5   Ibn Sīnā on inferences that are not syllogisms

From Ibn Sīnā *Burhān* i.8, 90.3–7 (my translation).

Your very knowledge that Zayd is a brother is your knowledge that he has a brother, or it is a component of your knowledge of that. But the conclusion [if you infer the latter from the former] shouldn't be something that is already as well known [to you] as the premise was. If it is not as well known, and your proving that he has a brother took place when you didn't already know it, then you didn't have in your mind the meaning of the sentence

$$\text{Zayd is a brother.} \tag{2}$$

Things like these are best not called syllogisms, let alone demonstrations.

1.6   Ibn Sīnā on making and discharging assumptions

From Ibn Sīnā *Qiyās* ix.3, 410.13–411.1 (my translation).

For example the usual way [to present an argument from absurdity] is it say

If [it's not the case that] not every C is a B, then every C is a B.
But every B is an A, so every C is an A, and this is absurd. $\quad$ (3)
Hence [not] every C is a B.

Thus when he says "so every C is an A", this means

If [it's not the case that] not every C is a B, then every C is an A. $\quad$ (4)

Thus if the circumstances are as we described, "then every C is an A". And his statement: "This is absurd" means

$$\text{Not every } C \text{ is an } A. \tag{5}$$

— which duplicates the contradictory of the second clause. So the usual style agrees with our analysis of the absurdity syllogism.

1.7   Ibn Sīnā on arguments remaining valid when condition is added

From Ibn Sīnā *Qiyās* ii.4, 109.16 (my translation).

When every C is a B
and every B is an A, $\quad$ (6)
then clearly every C is an A.

(Regarded as self-evidently valid.)

From Ibn Sīnā *Qiyās* vi.4, 326.3 and 331.14 (my translation).

Whenever p, then every C is a D.
Every D is an A. $\quad$ (7)
[It entails:] Whenever p, then every C is an A.

(Ibn Sīnā gives no proof.)

> Every C is a B,
> and whenever r then every B is an A.
> So whenever r, then every C is an A. $\qquad$ (8)

(Again no proof.)

From Ibn Sīnā *Qiyās* vi.5, 337.12f (my translation).

The first mood, where the condition is that the first clause is not impossible:

> Every C is a B, and
> whenever every B is an A, then r. $\qquad$ (9)
> It entails: It can be that when every C is an A, then r.

(Proof given but garbled.)

2 The formal system $\mathcal{IS}$ (Ibn Sīnā)

We introduce a proof calculus $\mathcal{IS}$ for first order logic, and we sketch a proof of its completeness. The calculus is based on techniques known to Ibn Sīnā, but I stress straight away that he would never have combined them in this form.

The language is a standard first-order language with truth-functions $\neg, \wedge, \vee$, quantifier symbols $\forall, \exists$ and infinitely many variables, but no identity. We assume the signature is relational and at most countable. We allow ourselves to add new variables at will.

The calculus is presented in the form of a set of sequents $T \vdash \phi$, where ϕ is a formula and T is a set of formulas. Some basic sequents are given outright, and there are also derivation rules for deriving sequents from other sequents. We describe these sequents with symbols ϕ, ψ etc. as metavariables for formulas, and x, y etc. as metavariables for variables. The valid sequents are those generated from these rules.

We write $T, \phi \vdash \psi$ for $T \cup \{\phi\} \vdash \psi$, and similar things. If x and y are variables, we write $\phi[y/x]$ for the result of replacing all free occurrences of x in ϕ' by y, where ϕ' is the result of replacing all bound occurrences of y in ϕ by occurrences of another variable y' distinct from y and not occurring in ϕ.

Basic sequents

(Refl) $\phi \vdash \phi$ $\qquad$ (NR)

(ExclM) $\vdash (\phi \vee \neg\phi)$

(NonC) $\vdash \neg(\phi \wedge \neg\phi)$

(ChrysL) $(\phi \vee \psi), \neg\phi \vdash \psi$ $\qquad$ (Syl)

(ChrysR) $(\phi \vee \psi), \neg\psi \vdash \phi$ $\qquad$ (Syl)

(DM∧) $\neg(\phi \wedge \psi) \vdash (\neg\phi \vee \neg\psi)$ $\qquad$ (Inf)

(DM∨) $\neg(\phi \vee \psi) \vdash (\neg\phi \wedge \neg\psi)$ $\qquad$ (Inf)

(DM∀) $\neg\forall x\phi \vdash \exists x\neg\phi$ $\qquad$ (Inf)

(DM∃) $\neg\exists x\phi \vdash \forall x\neg\phi$ $\qquad$ (Inf)

(∧I) $\phi, \psi \vdash (\phi \wedge \psi)$ $\qquad$ (Rec?)

(∧EL) $(\phi \wedge \psi) \vdash \phi$ $\qquad$ (Rec?)

(∧ER) $(\phi \wedge \psi) \vdash \psi$ $\qquad$ (Rec?)

(∀E) If t is any variable, then $\forall x\phi \vdash \phi[t/x]$ $\qquad$ (Inf?)

(∃I) If t is any variable, then $\phi[t/x] \vdash \exists x\phi$ $\qquad$ (Inf?)

(Vac) If x doesn't occur free in ϕ then $\exists x\phi \vdash \phi$ $\qquad$ (NR)

(Var) If y doesn't occur in ϕ then $\exists x\phi \vdash \exists y\phi[y/x]$ $\qquad$ (NR)

Derivation rules

(Mono) If $T \vdash \psi$ then $T \cup U \vdash \psi$.

(Trans) If $T \vdash \psi$ and for each $\phi \in T$, $U \vdash \phi$, then $U \vdash \psi$.

(IS) (Ibn Sīnā's Rule) Suppose T a set of formulas and ϕ, ψ are formulas. Let $\delta(p)$ be a formula of IS but containing a propositional variable p which occurs only positively in $\delta(p)$ and doesn't occur in the scope of any quantifier on a variable free in some formula of T. If $T, \phi \vdash \psi$ then $T, \delta(\phi) \vdash \delta(\psi)$.

We begin with some structural properties not depending on any symbol of the language.

Lemma 1 *If $T \vdash \phi$ is valid, then $U \vdash \phi$ for some finite $U \subseteq T$.*

Proof This is true for each of the basic sequents, and is preserved by the derivation rules. □

Lemma 2 *If $\phi \in T$ then $T \vdash \phi$.*

Proof Suppose $\phi \in T$. By (Refl), $\phi \vdash \phi$, and so by (Mono), $T, \phi \vdash \phi$. But $T \cup \{\phi\} = T$. □

Lemma 3 *If $T, \phi \vdash \psi$ and $T \vdash \phi$, then $T \vdash \psi$.*

Proof By Lemma 2, $T \vdash \chi$ for each $\chi \in T$. By this and the assumption $T \vdash \phi$, we have $T \vdash \chi$ for each $\chi \in T \cup \{\phi\}$. So $T \vdash \psi$ by (Trans) and the assumption $T, \phi \vdash \psi$. □

The proofs above illustrate the use of the rules (Refl), (Mono) and (Trans). In future we will normally use them without mention.

Now follow some basic properties of $\neg$ and $\vee$.

Lemma 4 $\neg\neg\phi \vdash \phi$

Proof By (ChrysR), $(\phi \vee \neg\phi), \neg\neg\phi \vdash \phi$. Now apply Lemma 3 with (ExclM). □

Theorem 1 (Deduction Theorem) *(a) If $T, \phi \vdash \psi$ then $T \vdash (\psi \vee \neg\phi)$.*
(b) If $T, \neg\phi \vdash \psi$ then $T \vdash (\psi \vee \phi)$.

Proof (a) Assume $T, \phi \vdash \psi$. By Ibn Sīnā's Rule (IS), $T, (\phi\vee\neg\phi) \vdash (\psi\vee\neg\phi)$. By (ExclM), $T \vdash (\phi \vee \neg\phi)$. Now the result follows by Lemma 3.
(b) Assume $T, \neg\phi \vdash \psi$. Then by (a), $T \vdash (\psi \vee \neg\neg\phi)$. But by Lemma 4 and another application of Ibn Sīnā's Rule (IS), $(\psi \vee \neg\neg\phi) \vdash (\psi \vee \phi)$. □

Lemma 5 $\phi \vdash \neg\neg\phi$

Proof By ($\wedge$I), $\phi, \neg\phi \vdash (\phi \wedge \neg\phi)$, and so by the Deduction Theorem 1(a), $\phi \vdash ((\phi \wedge \neg\phi) \vee \neg\neg\phi)$. So by (NonC) and (ChrysL), $\phi \vdash \neg\neg\phi$. □

Lemma 6 *(a)* $(\phi \vee \neg\psi), \psi \vdash \phi$
(b) $(\neg\phi \vee \psi), \phi \vdash \psi$.

Proof (a) By Lemma 5, $\psi \vdash \neg\neg\psi$, and by (ChrysR), $(\phi \vee \neg\psi), \neg\neg\psi \vdash \phi$.
(b) Similar with (ChrsL). □

Lemma 7 $(\phi \wedge \neg\phi) \vee \psi \vdash \psi$

Proof By (Contr) and (ChrysL). □

Lemma 8 $(\phi \vee \psi) \vdash (\psi \vee \phi)$.

Proof By (ChrysL), $(\phi \vee \psi), \neg\phi \vdash \psi$. Then by the Deduction Theorem, Theorem 1(b), $(\phi \vee \psi) \vdash (\psi \vee \phi)$. □

Lemma 9 $\phi \vdash \phi \vee \psi$.

Proof By (Refl), $\phi, \neg\psi \vdash \phi$. Now apply the Deduction Theorem, Theorem 1(b). □

Lemma 10 $(\phi \wedge \neg\phi) \vdash \psi$

Proof By Lemma 9, $(\phi \wedge \neg\phi) \vdash ((\phi \wedge \neg\phi) \vee \psi)$. Now apply Lemma 7. □

Lemma 11 *(a) If $\phi \vdash \psi$ then $\neg\psi \vdash \neg\phi$.*
(b) If $\neg\psi \vdash \neg\phi$ then $\phi \vdash \psi$.

Proof (a) Assume $\phi \vdash \psi$. Then $(\phi \vee \neg\phi) \vdash (\psi \vee \neg\phi)$ by Ibn Sīnā's rule (IS), so $\vdash \psi \vee \neg\phi$ by (ExclM). Now use (ChrysL), $(\psi \vee \neg\phi), \neg\psi \vdash \neg\phi$.
(b) Similar, via $(\psi \vee \neg\psi) \vdash (\psi \vee \neg\phi)$ and Lemma 6(a). □

Some quantifier lemmas:

Lemma 12 *Suppose x doesn't occur free in ϕ. Then $\phi \vdash \forall x\phi$.*

Proof By Lemma 11(b) it suffices to prove $\neg\forall x\phi \vdash \neg\phi$ under the same hypothesis. Now $\neg\forall x\phi \vdash \exists x\neg\phi$ by (DM$\wedge$), and $\exists x\neg\phi \vdash \neg\phi$ by (Vac), proving the lemma. □

Lemma 13 *Suppose x doesn't occur free in ψ or any formula of T, and $T, \phi \vdash \psi$. Then $T, \exists x\phi \vdash \psi$.*

Proof By Ibn Sīnā's Rule, $T, \exists x\phi \vdash \exists x\psi$. Then by (Vac), $T, \exists x\phi \vdash \psi$. □

Lemma 14 *Suppose x doesn't occur free in any formula of T, and $T \vdash \phi$. Then $T \vdash \forall x \phi$.*

Proof First suppose T is nonempty; let ψ be any formula of T. Then $T, \psi \vdash \phi$, so by Ibn Sīnā's Rule $T, \forall x \psi \vdash \forall x \phi$. Hence $T, \psi \vdash \forall x \phi$ by Lemma 12, and so $T \vdash \forall x \phi$ since $\psi \in T$.

If T is empty then let ξ be any formula not containing x, and reason as above to get $(\xi \vee \neg \xi) \vdash \forall x \phi$. Then the result follows by (ExclM). □

We say that a set T of formulas is *inconsistent* if for some ζ, $T \vdash \zeta$ and $T \vdash \neg \zeta$. If T is not inconsistent we say it is *consistent*.

Lemma 15 *If T is inconsistent then for every formula ξ, $T \vdash \xi$.*

Proof If $T \vdash \zeta$ and $T \vdash \neg \zeta$, then $T \vdash (\zeta \wedge \neg \zeta)$ by (∧I). It follows by Lemma 10 that $T \vdash \xi$. □

Lemma 16 *If T is consistent and $T \vdash \chi$, then $T \cup \{\chi\}$ is consistent.*

Proof Suppose that $T \vdash \chi$ but $T \cup \{\chi\}$ is inconsistent. Then for some ζ, $T, \chi \vdash \zeta$ and $T, \chi \vdash \neg \zeta$. Then $T \vdash \zeta$ by Lemma 3, and $T \vdash \neg \zeta$ for the same reason; so T is inconsistent. □

Lemma 17 *(a) If T is a set of formulas and ϕ a formula such that $T \cup \{\phi\}$ is inconsistent, then $T \vdash \neg \phi$.*
(b) If T is a set of formulas and ϕ a formula such that $T \cup \{\neg \phi\}$ is inconsistent, then $T \vdash \phi$.

Proof (a) Assume $T \cup \{\phi\}$ is inconsistent. Then by (∧I), there is some ζ such that $T, \phi \vdash (\zeta \wedge \neg \zeta)$. So by the Deduction Theorem, Theorem 1(a), $T \vdash ((\zeta \wedge \neg \zeta) \vee \neg \phi)$, and then $T \vdash \neg \phi$ by Lemma 7.
(b) Similar, using Theorem 1(b). □

Theorem 2 (Completeness Theorem) *If $T \models \phi$, where we regard free variables as constants, then $T \vdash \phi$.*

Proof In fact we will prove that every consistent set of formulas has a model. To derive the theorem as stated, suppose we don't have $T \vdash \phi$. Then by Lemma 17(b), $T \cup \{\neg \phi\}$ is consistent, so it has a model. This model is a counterexample to $T \models \phi$, so the theorem follows.

Now changing notation, we will show that if T is any consistent set of formulas, then we can extend T to a Hintikka set without losing consistency; we allow ourselves to add new variables to the language along the way. It's a standard result that Hintikka sets have models. I won't define Hintikka set, because the claims below make clear what the requirements are.

Claim 1 *Suppose T is consistent and $(\phi \wedge \psi) \in T$. Then $T \cup \{\phi, \psi\}$ is consistent.*

Proof of claim By (∧EL), (∧ER) and Lemma 16. □ Claim

Claim 2 *Suppose T is consistent and $(\phi \vee \psi) \in T$. Then at least one of $T \cup \{\phi\}$ and $T \cup \{\psi\}$ is consistent.*

Proof of claim Suppose to the contrary that $T \cup \{\phi\}$ and $T \cup \{\psi\}$ are both inconsistent. Then by Lemma 17(a) we have $T \vdash \neg \phi$. But by Lemma 2 and the assumption that $(\phi \vee \psi) \in T$, we also have $T \vdash (\phi \vee \psi)$, so $T \vdash \psi$ by (ChrysL). Now the same argument as for ϕ shows that $T \vdash \neg \psi$, which establishes that T is inconsistent. □ Claim

Claim 3 *Suppose T is consistent and $\neg(\phi \wedge \psi) \in T$. Then at least one of $T \cup \{\neg \phi\}$ and $T \cup \{\neg \psi\}$ is consistent.*

Proof of claim By (DM∧) and Lemma 16, $T \cup \{(\neg \phi \vee \neg \psi)\}$ is consistent. So the claim follows by Claim 2. □ Claim

Claim 4 *Suppose T is consistent and $\neg(\phi \vee \psi) \in T$. Then $T \cup \{\neg \phi, \neg \psi\}$ is consistent.*

Proof of claim By (DM∨) and Lemma 16, $T \cup \{(\neg \phi \wedge \neg \psi)\}$ is consistent. So the claim follows by Claim 1. □ Claim

Claim 5 *Suppose T is consistent and $\neg \neg \phi \in T$. Then $T \cup \{\phi\}$ is consistent.*

Proof of claim This is by Lemma 4 and Lemma 16. □ Claim

Claim 6 *Suppose T is consistent and $\forall x \phi \in T$. Then $T \cup \{\phi[t/x] : t \text{ a variable}\}$ is consistent.*

Proof of claim By Lemma 1 it suffices to show that we can consistently add a finite number of $\phi[t/x]$ to T; and we can show this by induction, adding one formula at a time. So it suffices to show that if t is any variable,

$T \cup \{\phi[t/x]\}$ is consistent. But we have this by ($\forall$E) and Lemma 16. $\square$
Claim

Claim 7 *Suppose T is consistent and $\exists x\phi \in T$, and let t be any variable not occurring in ϕ and not occurring free in any formula in T. Then $T \cup \{\phi[t/x]\}$ is consistent.*

 Proof of claim Let ξ be any formula in which t doesn't occur free. If $T \cup \{\phi[t/x]\}$ is inconsistent then by Lemma 15, $T, \phi[t/x] \vdash (\xi \wedge \neg\xi)$. So by Lemma 13, $T, \exists t\phi[t/x] \vdash (\xi \wedge \neg\xi)$. But by (Var), $\exists x\phi \vdash \exists t\phi[t/x]$, which shows that T is already inconsistent. $\square$ Claim

Claim 8 *Suppose T is consistent and $\neg\forall x\phi \in T$, and t has no free occurrence in $\forall x\phi$ or any formula in T. Then $T \cup \{\neg\phi[t/x]\}$ is consistent.*

 Proof of claim By (DM$\forall$) and Lemma 16, $T \cup \{\exists x\neg\phi\}$ is consistent. Then the claim follows using Claim 7. $\square$ Claim

Claim 9 *Suppose T is consistent and $\neg\exists x\phi \in T$. Then $T \cup \{\neg\phi[t/x] : t \text{ a variable}\}$ is consistent.*

 Proof of claim By (DM$\exists$) and Lemma 16, $T \cup \{\forall x\neg\phi\}$ is consistent. Then the claim follows using Claim 6. $\square$ Claim

 Together the claims show that if T is consistent, it can be extended to a Hintikka set, in general in a language with more variables. With this the proof is complete. $\square$

 The calculus for $\mathcal{SL}$ certainly contains some redundancies. For example we never used ($\exists$I). It can be derived from ($\forall$E), but almost certainly Ibn Sīnā would have regarded it as obviously correct in itself. By the same token, perhaps we should have included the results of Lemmas 4 and 5 as axioms, since Ibn Sīnā would certainly have reckoned that they are more obvious in themselves than their proofs are.