Tarski on Padoa's method

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`www.maths.qmul.ac.uk/~wilfrid/padoa.pdf`

Padoa's Method is a method for showing that a primitive notion R of a formal theory T is not definable in T from the remaining primitives of T.

Method: Give two models A and B of T which are identical except that $R^A \neq R^B$.

Example (Lindenbaum 1926): Let $\mathbb{R}$ be the real numbers, M the relation

$$M(a, b, c, d) \;\Leftrightarrow\; |a - b| = |c - d|$$

and P the relation 'a is strictly between b and c'.
Let T be the first-order theory of the structure $(\mathbb{R}, M, P)$.

We show by Padoa's method that P is not definable in T from M.

Define on $\mathbb{R}$:

$$a \sim b \;\Leftrightarrow\; |a - b| \text{ is rational .}$$

List the equivalence classes as $(C_i : i < \lambda)$ with $\lambda = 2^\omega$.
We define an increasing chain of bijections $\alpha_i : X_i \to Y_i$ where X_i and Y_i are unions of equivalence classes, so that each α_i is an M-isomorphism.

α_0 is the identity on $C_0 = X_0 = Y_0$.
Choose an irrational r, and define α_1 on $(C_0 \cup C_1) = X_1$ by

$$\alpha_1(a) = \begin{cases} a & \text{if } a \in C_0, \\ a + r & \text{if } a \in C_1. \end{cases}$$

α_1 is clearly a bijection extending α_0.
We show α_1 is an M-isomorphism.
Let $a, b, c, d \in C_0 \cup C_1$.

When $\alpha_i : X_i \to Y_i$ has been defined, we can define α_{i+1} on $X_i \cup C$, where C is a $\sim$-class disjoint from X_i, similarly.
For r we choose a real number not in the field generated by the numbers $|a - b|$ with $a, b \in X_i \cup Y_i$.

Alternatively we can define α_{i+1} with range $Y_i \cup C$, using the same construction in reverse.

We take unions at limit ordinals.
We write α for the union of the α_i.
Back-and-forth allows us to make α a permutation of $\mathbb{R}$.

If $a, b \in C_0$ and $c, d \in C_1$, then
$|\alpha_1(c) - \alpha_1(d)| = |(c + r) - (d + r)| = |c - d|$, so
$M(\alpha_1(a), \alpha_1(b), \alpha_1(c), \alpha_1(d) \Leftrightarrow M(a, b, c, d)$.

If $a \sim c$, $b \sim d$, $a < b$ and $d < c$, then $M(a, b, c, d)$ implies $b - a = c - d$, so

$$2a + r_1 = a + c = b + d = 2b + r_1.$$

Hence $a \sim b$, and we easily deduce
$M(\alpha_1(a), \alpha_1(b), \alpha_1(c), \alpha_1(d))$.

There are several other cases.

Define P' on $\mathbb{R}$ by:

$$P'(\alpha(a), \alpha(b), \alpha(c)) \;\Leftrightarrow\; P(a, b, c).$$

Then α is an isomorphism from $(\mathbb{R}, M, P)$ to $(\mathbb{R}, M, P')$.
So $(\mathbb{R}, M, P')$ is also a model of T.

But $P \neq P'$. Choose $b < c$ in C_0 with $|c - b| < r$, and a an element of C_1 between b and c. Then $\alpha(a) = a + r$ is not between $\alpha(b) = b$ and $\alpha(c) = c$.

Now apply Padoa.

MacTutor History of Mathematics Archive under ‘Padoa’:

> Tarski proved Padoa’s method in 1924.

In fact Tarski published in 1926 and 1935 papers claiming to give a ‘theoretical basis’ for Padoa’s method and to show its ‘generality’ (i.e. that it is necessary and sufficient).

Clearly something wrong here!

Tarski proved several variants of the following:

Let $\phi(X)$ be a formula with relation variable X, and R, S distinct relation constants not in ϕ.

Then in any reasonable second order logic
the following are equivalent:

(a) $\phi(R) \wedge \phi(S) \ \vdash \ R = S.$

(b) $\phi(R) \ \vdash \ \forall X(\phi(X) \leftrightarrow R = X).$

Besides being trivial, this result is purely syntactic.

Padoa presented this method at the 1900 International Congress of Philosophy in Paris.
He claimed that it is necessary and sufficient (for R to be undefinable in T from the remaining primitives).

‘Sufficient’ is certainly correct, given a reasonable logic.

‘Necessary’ is false. Padoa’s proof of necessity is the same as his proof of sufficiency, with a few words changed.

Counterexample: PA is second-order Peano arithmetic with full induction axiom and names for the natural numbers.

At most countably many sets of natural numbers are definable in the language of T'. Let X be a set that isn’t.

Let T be PA with added 1-ary relation symbol R and the sentences $R(n)$ (when $n \in X$) and $\neg R(n)$ (when $n \notin X$).

All models of PA are isomorphic by unique isomorphisms. So Padoa’s method can’t show R is undefinable in T.

‘Was Tarski a model theorist in the 1930s?’

A nonsense question.
In the 1930s there was no such thing as a model theorist.

We need to ask questions that make sense in context.

- In the 1930s Tarski used methods that we now count as model-theoretic.
- In the 1930s Tarski believed that these methods are *not mathematical*. They are ‘metamathematical’ or ‘methodological’.
- For us this distinction is uninteresting.
  For Tarski in the 1930s it was the most important thing in the world.

So Tarski’s condition (a) is necessary and sufficient for definability of R in second order logic,
but (a) is not Padoa’s method.

In 1935 Tarski refines his result to get a necessary and sufficient condition for definability of R in terms of a given subset of the primitives of ϕ.
The condition is again second-order and purely syntactic.

Wilfrid Hodges, ‘Truth in a structure’, *Proceedings of Aristotelian Society* 86 (1985/6):
In the 1930s Tarski didn’t have the model-theoretic truth definition.

Wilfrid Hodges, ‘What languages have a Tarski truth definition?’, *Annals of Pure and Applied Logic* 126 (2004):
We can chart Tarski’s gradual progress from the 1933 truth definition to the model-theoretic one which he found in 1951/2, as he responded to technical needs.

For Tarski in the 1930s, to give a foundation for Padoa's method is to *remove the model theory* and replace it by calculations within the deductive system of the theory T.

His papers give a possible format
for the calculation within the deductive system,
but they don't explain how we actually convert
from model theory to deductive system.

We can convert Lindenbaum's example above to a calculation done with a higher-order theory T of $(\mathbb{R}, M, P, W)$ where W is a well-ordering of $\mathbb{R}$.

In this theory we can explicitly define α and P'.
We can show that for each formula ϕ, if ϕ' is ϕ with P replaced by P',

$$T \;\vdash\; \phi(a_1, \ldots, a_n) \to \phi'(\alpha(a_1), \ldots, \alpha(a_n))$$

by induction on the complexity of ϕ. This is a *syntactic* metatheorem, not using any model theory.

For Tarski in the 1930s a *deductive system* is a form of activity.
We 'practise' (*uprawiać*) formal systems.
(Compare *uprawiać bezpieczny seks*, 'practise safe sex'.)

We practise a deductive system by writing down a logic and axioms in the language of the logic, and deducing propositions from the axioms by the rules of the logic.

Tarski's teacher Leśniewski wrote papers doing exactly this.
(They are totally unreadable.)

For Tarski in the 1930s, *mathematics* is what we can do in deductive systems (though often we do it informally).
Metamathematics (or *methodology*) is the study of deductive systems.

Carnap:

> "Tarski came to Vienna in 1930. ... Of special interest to me was his emphasis that certain concepts used in logical investigations ... are to be expressed not in the language of the axioms ... but in the metamathematical language."

Today almost nobody sees model theory in these terms.

Hence Tarski's early work is constantly misread.

If interpreting Polish logic of 1930 is so hard,
how much more should we honour Bilal Krishna Matilal
for decoding the Indian logic of many centuries ago
and interpreting it to people of another culture.

We can deduce that all of T' (i.e. T with P replaced by P')
follows from T.
Also we can prove $P \neq P'$ from T.
Hence assuming the consistency of T,
$P = P'$ is not provable from $T \cup T'$.

So by Tarski's version of Padoa,
P is not definable from M.

This rewriting of Lindenbaum's argument uses no
model theory at all.

I suspect it is close to what Lindenbaum actually did.

Tarski probably regarded the explanation in terms of
'interpretations' as an informal description of the argument,
not an argument in its own right.