Why modern logic took so long to arrive: Three lectures

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A potted history in five periods

1. The classical Greeks, 4th century BC

Aristotle introduces the idea of checking arguments by showing that they conform to valid argument patterns.

He describes a systematic collection of valid argument patterns (syllogisms) and some sporadic ones (topics).

Chrysippus describes some propositional argument patterns.

2. The Roman Empire, 2nd to 6th centuries AD

Aristotle's works, edited by Andronicus of Rhodes, become the basis of liberal education.

Late 3rd century, Porphyry protects Aristotle's logic from ideological disputes by separating it from metaphysics and proposing a basis in terms of meanings.

Details worked out by his followers and the Alexandrian school of Ammonius.

6th century, Boethius paraphrases in Latin Porphyry's views on logic.

Lecture One: What traditional logic covered

and in particular the place of relational reasoning

3. The Arabs, 8th to 13th centuries

Excellent translations made of Aristotle. Translations of Roman Empire commentators become available in libraries of connoisseurs as far east as Afghanistan.

10th century, Al-Fārābī writes major commentary (now mostly lost) on Aristotle's logic.

11th century, Ibn Sīnā an independent thinker in the aristotelian tradition (compare Leibniz and Frege — as we will). One of his textbooks of logic is over 2000 pages.

Later writings on logic (e.g. Ibn Rushd 12th century, Tusi 13th century) show less independence.

4. The Scholastics, 12th to 15th centuries

12th century, Abelard virtually reinvents logic on basis of Boethius and an incomplete set of texts of Aristotle.

12th to 13th centuries, Terminists develop theory of supposition. (Unclear whether it's about the notion of reference or about infinitary proof rules.)

Early 14th century, Jean Buridan probably the most technically proficient of the Scholastics.

Throughout this period, theory of logic was confined to a handful of universities.

5. Renaissance and Enlightenment, 15th to 19th centuries

Logic emerges from the universities and becomes education for barristers and young ladies.

17th century, attempts to adapt logic to the spirit of the age of Descartes and Newton. Main figures Arnauld and Nicole (Port-Royal Logic) and above all Leibniz.

19th century, old idea of logic as protection against error revived in more sophisticated form, in particular by Frege.

With Peano (1890s) the connection to Aristotle is finally lost.

Aristotle's syllogisms

A typical syllogism, as Aristotle wrote it (*Prior Analytics i.6*):

> If R belongs to every S, and P to no S, there will be a deduction that P will necessarily not belong to some R.

Here 'R', 'P', 'S' stand for nouns. (Proper names are treated as common nouns.) Thus for example:

> If every animal is mobile, and no animal is eternal, then necessarily some mobile thing is not eternal.

Hilbert used this topic as an argument for adopting first-order logic:

> "If there is a son, then there is a father," is certainly a logically self-evident assertion, and we may demand of any satisfactory logical calculus that it make obvious this self-evidence, in the sense that the asserted connection will be seen, by means of the symbolic representation, to be a consequence of simple logical principles.

This is from Hilbert's Göttingen lectures of 1917–1922, published in 1928 in his textbook with Ackermann.
In these lectures Hilbert created the syntax and semantics of first-order logic.

What's so hot about first-order logic, if this topic was already known to Boethius in the 6th century?

Answer (in Hilbert's text): First-order logic is a logical calculus that analyses *both* syllogisms *and* relational arguments like this one down to 'simple logical principles'.

Later logicians saw that to make this example work, we need to assume something about existence.

The usual assumption was that if there are no Ss then

- 'Some S is an R' is *false*;
- 'Every S is an R' is *false*;
- 'Some S is not an R' (negation of 'Every S is an R') is *true*;
- 'No S is an R' is *true*.

Topics

Also known as *places*. The ones for use in general situations are called *common places*; Renaissance scholars suggested keeping a *commonplace book* for them. The most general topics are called *maximal places*, abbreviated to *maxims*.

Example:

> If one of the correlated things is posited, the other is posited. (Peter of Spain, *Tractatus* V para. 28.)

This is the pattern behind the inference

> If there is a father then there is a child.

Some Basics

How are syllogisms supposed to be used, in practice, for validating arguments?

There are some scattered remarks about this at the end of Aristotle's *Prior Analytics* i.

They show that Aristotle's practice was probably almost identical to modern elementary logic courses.

Given an argument in Greek, Aristotle shows that it fits some valid syllogism by showing what terms in the argument correspond to the term symbols in the syllogism. E.g. (*Prior Analytics* i.35):

A	:	having internal angles that sum to two right angles.
B	:	triangle.
C	:	isosceles triangle.

He calls this *setting out the terms*.

For the rest of this lecture, we ask why nobody before the end of the 19th century seriously tried to integrate syllogisms and relational topics into a calculus covering both.

One meets three views:

- (Joachim Jungius) Relational topics are *sui generis*.
- (Leibniz) Relational topics can be reduced to syllogisms plus paraphrasing.
- (Buridan, De Morgan) Syllogisms and some relational topics are justified by a higher-level principle called *dictum de omni et nullo*.

Jungius is chiefly memorable for having stimulated Leibniz. The other views, plus relevant opinions of Ibn Sīnā, all revolve around the question whether one can really reason with anything beyond syllogistic sentences

(Some/Every) A (is/isn't) a B,

where the terms in place of A and B are taken as impenetrable.

The view that one can't is what I call *Top-Level Processing*, TLP for short. (Some refinements will follow.)

Natural language paraphrase:

> No (time needing to be set aside for action) is a (thing that God has). Some (right moment for action) is a (thing that God has). Therefore some (right moment for action) is not a (time needing to be set aside for action).

The subject and predicate terms are marked with brackets. I refer to the corresponding parts of the original argument as the *eigenterms* (adapting Gentzen).

Why does the paraphrase of each sentence have to be a syllogistic sentence?

From a modern perspective, a no-brainer. We are validating by syllogisms, and this just is the form of the sentences in syllogisms. In another logic, other forms would be used.

But traditional logicians often seem to want to show that reasoning *can only use* syllogistic (or very similar) sentences. This blocks any attempt to find a more powerful logic.

Setting out terms (in this sense) disappears after Aristotle and reappears only in Boole 1847. What takes its place?

To check an argument, we have to verify that (for example) the first premise means the same as 'Every B is an A', where 'A', 'B' are interpreted as in the setting out of terms.

Instead, the traditionals rewrote 'Every B is an A' using the expressions in the setting out of terms, and checked that the result means the same as the premise.
So they made a *natural language paraphrase* of the premise.

Example (Aristotle, *Prior Analytics i.36*)

> God doesn't have times that need to be set aside for action. God does have right moments for action. Therefore some right moment for action is not a time that needs to be set aside for action.

Setting out:
A : thing that God has.
B : time needing to be set aside for action.
C : right moment for action.

Syllogism:
No B is an A. Some C is an A. Therefore some C is not a B.

Leibniz believes that the key to reasoning is that we can paraphrase the premises so as to bring the eigenterms to nominative case.

Opuscules p. 287:

> A reading of poets is an act by which a poet is read. . . . Paris is a lover of Helen, i.e. Paris is a lover, and thereby Helen is loved. So there are two propositions packed into one. . . . If you don't resolve oblique cases into several propositions, you will never avoid being forced (like Jungius) to devise new-fangled ways of reasoning.

Remark

Leibniz's basic strategy here is that of Peirce 'The reader is introduced to relatives' (1892) and Davidson 'The logical form of action sentences' (1967):

> Quantify over actions or events, and treat other parts of the situation as attached to the action or event in standard ways (e.g. as AGENT or OBJECT).

This doesn't get rid of relations, but it limits them to standard ones built into the language.

Ibn Sīnā has a complex and highly integrated theory that uses logic to explain the workings of the rational mind, including the kinds of error that one can make.

When we realise for the first time that something is true, on the basis of a rational argument, this is because our minds analyse the data, notice common features between the propositions expressing the data, and by identifying these features, produce a new combination of ideas.

'Recombinant' syllogisms (Aristotle's syllogisms and their adaptations to propositional logic) express this kind of argument.
Our minds are a kind of PROLOG inference engine.

In short, the basic ingredient of reasoning is to bring together two ideas.
The syllogistic sentences express such a union of ideas.
Hence Top-Level Processing.

Why the nominative? (It's not clear it actually helps in Leibniz's example above.)

Buridan (*Summulae* 4.2.6) gives various arguments why the linguistic subject and predicate of a subject-predicate sentence need to be in the nominative. In *Summulae* 5.8.2 he deduces that linguistic subjects and predicates are not the same thing as eigenterms.

But he leaves it open that eigenterms can all be brought to the nominative by *paraphrase*. This could be a TLP justification for Leibniz's use of nominative.

De Morgan

Formal Logic p. 114f:

> 'man is animal, therefore the head of a man is the head of an animal' is inference, but not syllogism. . . . there is a postulate which is constantly applied. . . . It contains the *dictum de omni et nullo* (see the next chapter), and it is as follows. For every term used universally *less* may be substituted, and for every term used particularly, *more*.

Leibniz's proof that 'If (every kind of) painting is an art, then a person who learns (some kind of) painting learns an art':

(1) A person who learns painting learns a thing which is painting.

(2) Therefore a person who learns painting learns a thing which is an art.

(3) A person who learns a thing which is an art learns an art.

(4) Therefore a person who learns painting learns an art.

In Latin, steps (1) and (3) bring 'painting' and 'art' to the nominative.

In fact Leibniz's unpublished papers contain all the techniques needed to prove this inference, but the nearest he comes in print is from his posthumous book:

> It should also be realized that there are *valid non-syllogistic inferences* which cannot be rigorously demonstrated in any syllogism unless the terms are changed a little, and this altering of the terms is the non-syllogistic inference. There are several of these, including arguments from the direct to the oblique — e.g. 'If Jesus Christ is God, then the mother of Jesus Christ is the mother of God'. (*New Essays* 479f.)

Frege in *Begriffsschrift* 1879 goes to the heart of the matter: we need proof rules like

$$\phi(s),\ s = t \ \vdash\ \phi(t)$$

which apply arbitrarily deep in a formula.

But curiously there are two signs of previous history in his account of this.

1. His picture is not that s can be arbitrarily deep in ϕ, but that ϕ has a movable top level.

2. One special case of upwards monotonicity is the rule of modus ponens applied to positively occurring subformulas:

$$\frac{\phi(\theta) \qquad\qquad \forall\bar{x}(\theta \to \eta)}{\phi(\eta)}$$

Frege sets up the language of *Begriffsschrift* so that it's particularly easy to spot positively occurring subformulas. I think nobody since Frege has taken this seriously.

We say an occurrence of a term A in a sentence $\phi(A)$ *allows downward monotonicity* if $\phi(A)$ and 'Every B is an A' together entail $\phi(B)$; likewise *upward monotonicity* with 'Every A is a B'.

De Morgan's claim is that $\phi(A)$ allows downward (resp. upward) monotonicity if A is universally (resp. existentially) quantified in ϕ.

This is badly garbled — what does 'used particularly' mean?

It could be corrected (and would then extend syllogisms to some relational arguments, but not many) if we replace 'used universally' by 'occurring negatively', and 'used particularly' by 'occurring positively'.

De Morgan himself shows no awareness of this correction. Buridan (*Summulae* 5.8.2) also invokes monotonicity, but his criterion for it seems to be question-begging.

From Ibn Sīnā's *Autobiography* (Gutas' translation):

> I read Logic and all the parts of philosophy once again. … I compiled a set of files for myself, and for each argument that I examined, I recorded the syllogistic premisses it contained, the way in which they were composed, and the conclusions which they might yield, and I would also take into account the conditions of its premisses [i.e. their modalities] until I had Ascertained that particular problem. … Having mastered Logic, Physics and Mathematics …

This shows that Ibn Sīnā reckoned to validate an argument by validating *each inference step separately*.
Call this *local formalisation*.

Recall Leibniz's description of the non-syllogistic steps that consist of paraphrases.
These alternate with the syllogistic steps.

The modern approach, since Frege and Peano, is to formalise an argument *globally*.

Lecture Two: Discharge of assumptions

The major scandal of the history of logic

Traditional logicians report that logic contains all the methods of argument of all the exact sciences.
They never express any doubt about this.

Among traditional logicians there were many excellent mathematicians: Tusi, Leibniz, Wallis, John Bernoulli, Gergonne, De Morgan for example.

But nobody today believes that traditional logic is remotely adequate to formalise the *Elements* of Euclid. *Explain!*

Since Prawitz, natural deduction systems usually have four rules that ‘employ the means of an arbitrary supposition’.

1. →-Introduction

$$\frac{\begin{array}{c}[\phi]\\ \vdots \\ \psi\end{array}}{\phi \to \psi}$$

2. Reductio Ad Absurdum (RAA)

$$\frac{\begin{array}{c}[\neg\phi]\\ \vdots \\ \bot\end{array}}{\phi}$$

Many things are easier if one only formalises locally.
But some are harder, because each step has to be self-supporting.

In particular we can’t make an assumption in one step and discharge it in another.
So natural deduction rules in full generality could only come after Frege and Peano.

Jaśkowski 1934:

> In 1926 Professor J. Lukasiewicz called attention to the fact that mathematicians in their proofs do not appeal to the theses of the theory of deduction, but make use of other methods of reasoning. The chief means employed in their method is that of an arbitrary supposition. The problem raised by Mr Lukasiewicz was to put those methods under the form of structural rules and to analyse their relation to the theory of deduction.

For example Proclus, *Commentary on Euclid's Elements I* 255f:

> (*Theory*) Every reduction to impossibility takes the contradictory of what it intends to prove and from this as a hypothesis proceeds until it encounters something admitted to be absurd and, by thus destroying its hypothesis, confirms the proposition it set out to establish.

Proclus continued:

> (*Example*) For example, if in triangles that have equal angles the sides subtending the equal angles are not equal, the whole is equal to the part. But this is impossible; therefore in triangles that have two angles equal the sides that subtend these equal angles are themselves equal.

3. ∃-Elimination

$$\frac{\exists x\phi \qquad \begin{matrix}[\phi(a/x)]\\ \vdots \\ \psi\end{matrix}}{\psi}$$

where a doesn't occur in ψ, ϕ or any assumption apart from $\phi(a/x)$.

And a similar rule of $\vee$-Elimination.

In Aristotle and the Roman Empire period, RAA is said to start by assuming the contradictory of the goal (the thing one wants to prove at the end), and deducing a contradiction.

In examples given by early logicians, the 'assumption' always turns out to be the antecedent of a conditional.

The same words ('hypothesis', 'posit') are used for both assumptions and antecedents.

Ibn Sīnā *Qiyās 8.3*:

> The usual way to present a reductio ad absurdum is like this:
> *If [not] not every J is B, then every J is B.*
> *But every B is A.*
> *So every J is A, contradiction . . . .*
> *Hence [not] every J is B.*
> When he says "so every J is A", this means
> *If [not] not every J is B, then every J is A.*

Ibn Sīnā is saying that people often introduce the hypothesis as an antecedent, but then don't repeat it.
(So the hypothesis is an assumption by criterion (b).)
Many examples confirm Ibn Sīnā's observation, e.g. Alexander.

His repair is to add the hypothesis as an antecedent all the way down.

We will see that Frege does the same.

Burley, *De Puritate Logicae*, 14th c AD, compares three forms:

(i) If A then B. (*Si* A, B.)

(ii) With respect to A, B follows. (*Ad* A *sequitur* B.)

(iii) A, therefore B. (A *ergo* B.)

(i) is a conditional, (ii) and (iii) are 'consequences' that report arguments.
Burley says (i) performs the act that is signified by (ii).
But he says exactly the same of (ii) and (iii).

He has no absolute distinction between assumptions and antecedents.

Differences between assumptions and antecedents

(a) Assumptions are introduced with 'Let' or 'Suppose', antecedents with 'If'.

(b) Assumptions only need to be repeated when they are used. Antecedents must be repeated at every step.

(c) Assumptions must be discharged, antecedents can't be.

By (a), Euclid has both assumptions and antecedents.

Port-Royal Logic (Arnauld and Nicole 1662) iii.13:

> Example. If I want to prove that the moon is a rough-surfaced body . . . , I need three propositions to show this absolutely.
>
> *Every body that reflects light from all parts is rough-surfaced;*
> *The moon reflects light from all parts;*
> *Therefore the moon is a rough-surfaced body.*

> But I need only two propositions to show this conditionally, as follows:
>
> *Every body that reflects light from all parts is rough-surfaced;*
> *Therefore if the moon reflects light from all parts, then it is a rough-surfaced body.*
>
> . . . This style of reasoning is very common and very beautiful.

They note that the second form removes any requirement to accept the second premise; i.e. it is discharged.

If the assumption is not discharged, then the conclusion depends on it.

But there is another way of avoiding discharging it: Count the part of the argument that depends on the assumption as a separate argument which is *mentioned* but not *used*.

So we never actually make the assumption; we just use the fact that there is an inference which makes it.

This device appears in Buridan, for example. But it's incompatible with global formalising.

The first place known to me where assumptions are explicitly discharged is the *Port-Royal Logic* of Antoine Arnauld and Pierre Nicole, 1662.

The assumption is cancelled by being added to the conclusion as antecedent.

This is exactly →-Introduction. But it's for a single syllogism. Thanks to local formalising, we shouldn't expect more.

Ibn Sīnā's analysis

Suppose we have a valid entailment (A):

$$\frac{p \qquad q}{r}$$

Then the following entailment (B) is also valid
(where we write $\bar{p}$ for the contradictory negation of p):

$$\frac{\bar{r} \qquad q}{\bar{p}}$$

Next, adding a contradiction at the bottom yields
a two-step syllogism (C):

$$\frac{\dfrac{\bar{r} \qquad q}{\bar{p}} \qquad p}{\bot}$$

The final step (D) is the touch of genius. Ibn Sīnā adds the antecedent $\bar{r}$ to all the lefthand formulas all the way down. At the bottom, $\bar{r} \to \bot$ is equivalent to r, which was the desired conclusion.

$$\frac{\dfrac{\bar{r} \to \bar{r} \qquad q}{\bar{r} \to \bar{p}} \qquad p}{r}$$

(How is the addition of $\bar{r}$ justified? It's a kind of fibration.
Ibn Sīnā seems to be following intuition here.)

In *Über die Grundlagen der Geometrie* Frege makes a further point.
Assumptions often introduce a constant without giving it a reference.

See ∃-Elimination. Also there's a near miss in Alexander's reductio example. The numbers E and F are in fact uniquely determined, though the argument doesn't need this.

Such assumptions don't state anything,
so they can't be either true or false.
If ϕ is such an assumption, then to make a derivation of ψ from ϕ into a Schluss, we need to add ϕ as antecedent everywhere, *and then universally quantify*.

Thus with one pen stroke Frege shows the complete irrelevance of counterfactual conditionals for understanding reductio ad absurdum.

Remark

In the top left syllogism, the premise $\bar{r} \to \bar{r}$ is a tautology and can be removed.
The result is to convert (B) as follows:

$$\frac{\bar{r} \qquad q}{\bar{p}} \qquad \Rightarrow \qquad \frac{q}{\bar{r} \to \bar{p}}$$

This is almost →-Introduction.

In his *Logik in der Mathematik* (unpublished 1914) Frege repeats the main points we quoted from the *Port-Royal Logic*, but without the restriction to a single inference step.

He also distinguishes between a derivation (*Ableitung*), which is a logically valid sequence of inferences, and a demonstration (*Schluss*) which shows that something is true by deriving it from things already known to be true.
An Ableitung can be converted to a Schluss by turning the assumed premises into antecedents.

This passage has been attacked by several authors recently, I think because the difference between Ableitung and Schluss is invisible in the English translation.

Lecture Three: Type-theoretic semantics

For about two thousand years, study of language has been divided into:

Semantics — what words and sentences mean.

Syntax — parts of speech (Dionysius Thrax, 2nd c BC), agreement and grouping (Apollonius Dyscolus, 2nd c AD).

Strong linguistic traditions in other cultures (particularly Indian) had little influence.

To reach his Schluss, Frege asks ‘What do we know as a result of giving this derivation?’

Ironically, in his reworking, each step is complete and self-contained in the way required by local formalising.

Analogues of Frege’s moves appear in soundness proofs for natural deduction calculi. So his position about how to understand these calculi is broadly vindicated.

Finally, did Ibn Sīnā have enough logic to validate all the sound arguments in Euclid’s *Elements*?

My guess, based on more evidence than I can report here, is that in principle he did, but the techniques would have needed a lot of polishing first.
I also guess that this was his hunch too.

Richard Montague (Tarski's student): The next step is to build up semantics of natural languages, using their syntax as template, just as Tarski did for formal languages.
Heim and Kratzer, *Semantics in Generative Grammar*, Blackwell 1998.

This involves assigning a type structure and giving meanings that fit the types.

All very recent.

In the aristotelian tradition, at least till the Renaissance, semantics was in the hands of the logicians.

The late 13th century Modist attempt to build a semantics based on grammar was a catastrophe.
Thomas of Erfurt, *Grammatica Speculativa* §8, §32:
The 'genitive mode of signifying' is derived from the 'genitive mode of being' via the 'genitive mode of understanding'.

(Quia est in eo virtus dormitiva.)

Rubbished for all the wrong reasons by Ockham etc. in the 14th century.

Until 1930s, everyone regarded syntax as built on semantics. Language is for conveying meanings, and syntax is how a particular language wraps up the meanings.

Bloomfield, *Language* (1933) p. 138:

> A phonetic form which has a meaning, is a *linguistic form*.

His italics. His notion of 'constituent' rests on this definition.

A major shift came with Noam Chomsky around 1960:
The syntax of a language is generated independently of the semantics, and can be studied independently of it.

Probably Tarski's 1930 truth definition, channelled through Quine, was a major influence behind this.

Two barely consistent accounts of the top left arrow:

1. The arrow expresses the speaker's intention

Diodorus Cronus (in Aulus Gellius *Noctes Atticae* 11.12):

> [A word] ought not to seem to be said in any other sense than that which the speaker feels that he is giving to it.

The 12th century translators of Ibn Sīnā's metaphysics into Latin translated his word for 'meaning' as 'intentio'. Hence *intentionality*.

The 'speaker's intention' view survived till Hilbert (1899 letter to Frege):

> Wenn ich unter meinen Punkten irgenwelche Systeme von Dingen, z.B. das System: Liebe, Gesetz, Schornsteinfeger . . . **denke**

It was vigorously attacked by Frege as subjective.

It was eventually replaced by Tarski's semantics, which makes the interpretation of symbols a set-theoretic function.

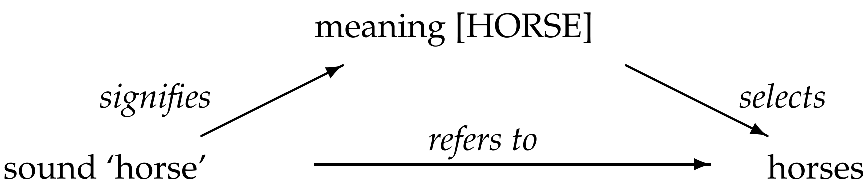
From Roman Empire times, logicians developed semantics to support syllogistic reasoning.

Andronicus of Rhodes (1st c BC) edited Aristotle's works, putting in order:

- *Categories* (single nouns),
- *De Interpretatione* (constructions used in definitions and syllogisms),
- *Prior Analytics* (syllogisms).

This encouraged a bottom-up semantics.

The De Interpretatione triangle



Our notion of 'word' consists of the top left arrow.

On the first question: Porphyry says (*Categories* 56.12) a noun signifies

> being a such-and-such.

I.e. it signifies a way of classifying objects into those which are such-and-such and those which aren't.
He is conspicuously noncommittal about further details.

A meaning of this kind is in modern terms of type $(e \to t)$, where $e =$ type of objects, $t =$ type of truth values.
So it's not at bottom level.

Bottom level e should be the type of proper names.
But Basil of Caesarea (4th c, in Sorabji *Sourcebook* III p. 227):

> The names [of particular men] are not actually signifiers of substances, but of the distinctive properties which characterise the individual.

Basil may be relying on earlier Stoic sources, but this view was accepted by Porphyry and his successors.
It had the effect of preventing the use of type e until Frege.

2. The arrow expresses the imposition of the sound on the meaning at the beginning of language

This account has religious authority (*Genesis* 2.19):

> And out of the ground the LORD God formed every beast of the field, and every fowl of the air; and brought them unto Adam to see what he would call them: and whatsoever Adam called every living creature, that was the name thereof.

Likewise in the Qur'an: Allah taught Adam the meanings of all nouns.

The imposition account begs two obvious questions.

- How does tying a word to one or more individual objects determine what *other* objects the word applies to?
- How does imposition work for other kinds of word, such as proper names, quantifiers, conjunctions etc.?

Porphyry (late 3rd century) establishes a programme to answer these questions *so far as needed for logic*.

There remain two major questions:

- How to give the meanings of *particles* ‘in’, ‘every’, ‘is’ etc., none of which select a class of objects.
- How to construct the meaning of a compound phrase from those of its constituents.

These two questions converge, because by classical linguistic theory, particles work by forming compounds.

Ibn Sīnā (discussing semantics):

> It’s a black mark against Aristotle that he mentions among the simple expressions the noun and the verb, but ignores the particles. (^{c}*Ibāra* 29.15)

So Ibn Sīnā reckons this question has been answered by logicians since Aristotle.
He probably means Porphyry’s school.

Likewise type t was hidden by taking the type of sentences to be the type $(s \to t)$ of *kinds of situation*, except where quantified out.
Ibn Sīnā, Wallis and others took this for granted.

Leibniz was uncommitted on it.
Maybe he saw the problem:
How do we get type $(s \to t)$ for ‘Madonna is a singer’ when ‘Madonna’ and ‘singer’ both have type $(e \to t)$?

John Wallis, *Institutio Logica* 1702 p. 288:

> ‘If the sun is shining, it’s day. The sun is shining.
> Therefore it’s day.’
>
> [This can be rewritten]
>
> ‘In every place where the sun is shining, it’s day.
> The sun is shining somewhere.
> Therefore it’s day somewhere. ’

The motive is to reduce propositional logic to syllogistic.

Modern example (Colin Stirling, *Modal and Temporal Properties of Processes* p. 2):

> Behaviour of processes is captured by transitions $E \xrightarrow{a} F$, that E may evolve to F by performing or accepting the action a. ...
> A process $a.E$ performs the action a and becomes E.

The analogous statement in Ibn Sīnā's situation is that 7 loses 1 and becomes 6.

Not a criticism of Stirling;
his metaphor is helpful and well understood.
But until Euler in the 18th century there were *only metaphors* for this situation.

Frege attacks the same metaphor (*Was ist eine Funktion?* 1904):

> The number 1,000 has not somehow swollen up to 1,001, but has been replaced by it. Or is the number 1,000 perhaps the same as the number 1,001, only with a different expression on its face?

Ibn Sīnā's theory of functions

We analyse the following remark (Ibn Sīnā ^{c}*Ibāra* 15.9ff):

> There is a common kind of error about things that are joined together. It occurs through not recognising that an idea taken with another idea is not the whole arising from it and the thing taken with it; just as one added to six, when we consider it together with six, is not the sum of one and six, which is seven.

This comes in a discussion of *compound phrases*. He clearly thinks combining meanings is like adding numbers.

For Ibn Sīnā, the meaning of '$1 + 6$' is got by combining the meanings of '1', '6' and '$+$'.
The first two are proper names, hence covered by the remarks above.

The third is a particle. It signifies how 1 is to be linked to 6, i.e. how we get from 6 to $6 + 1$.

Why does Ibn Sīnā think people confuse 6 with $6 + 1$?

Answer: Because they do; they say 6 *becomes* 7.

and Frege (*Funktion und Begriff* 1891):

> Statements in general, just like equations or inequalities or expressions in Analysis, can be imagined to be split up into two parts; one complete in itself, and the other in need of supplementation, or 'unsaturated'. Thus, e.g., we split up the sentence 'Caesar conquered Gaul' into 'Caesar' and 'conquered Gaul'. ... I give the name 'function' to what this 'unsaturated' part stands for.

In particular the meaning of a particle P is (for Ibn Sīnā) incomplete. How should the meaning of P be defined?

The answer is in Ammonius c. 500 AD.

We give the meaning of a sentence $P(X)$,
in a way which shows how the meaning of $P(X)$ depends on the meaning of X.

In Frege's version (*Grundlagen der Arithmetik* 1884, §46, tr. Jacquette) we consider the particle 'in the context of a judgment where its primary method of application is prominent'.

Ibn Sīnā's observation is correct for the whole aristotelian tradition, including himself.

But he has the beginnings of a theory of functions, identical with Frege's as far as it goes.

Frege's 'function' is Ibn Sīnā's 'part of compound which is incomplete until attached to the rest of the compound'.

Frege (*Was ist eine Funktion?*):

> Here we come upon what distinguishes functions from numbers. '*Sin*' requires completion with a numeral.

Compare Ibn Sīnā (*Ishārāt*, tr. Inati p. 52 with adjustments):

> Examples of an incomplete phrase are "in the house" and "not a human being". A part of [expressions] such as these two is intended to have signification, but one of the two parts, such as "not" and "in", is a particle whose meaning is not completed unless linked [to another term].

(Note the Stirling metaphor!)

Ibn Sīnā comes close to extending the 'incomplete meaning' account to participles, because they have an 'indeterminate agent'.

> For example the word "walking". It signifies the act of walking and the indeterminate agent, and that the act belongs to the agent. (^{c}*Ibāra* 18.9f)

He believes that the participle inherits the 'indeterminate agent' from the verb.

If Ibn Sīnā had extended this from participles to all common nouns, he would have reached Frege's position in the second Grundsatz of the *Grundlagen der Arithmetik*:

> Seek the meanings of words in the interconnections of the sentence, not in the words taken independently.
> Nach der Bedeutung der Wörter muss im Satzzusammenhange, nicht in ihrer Vereinzelung gefragt werden.

Example: Ammonius' explanation of the meaning of 'Every' (*De Interpretatione* 89,4ff).

> Determiners ... combine with the subject terms and indicate how the predicate relates to the number of individuals under the subject; ...
> 'Every man is an animal' signifies that 'animal' holds of all individuals falling under 'man'.

Compare Bertrand Russell *On Denoting* 1905 (who thought this was his discovery):

> *everything* [is] to be interpreted as follows:
>
> C(everything) means '$C(x)$ is always true'.

The form of the definition depends on the type of the particle: $(x \to t)$ if adding a phrase of type x gives a sentence. Frege was the first to spell this out for specific types, in *Grundgesetze der Arithmetik* I §33 (1893).

Then it went underground until Tarski gave the definitive account for first order relational languages in *Undecidable Theories* I.4, 1953.
Thus a binary function symbol $+$ requires a definition

$$\forall x \forall y \forall z (x = y + z \leftrightarrow \phi(x, y, z))$$

with an obvious requirement on ϕ.

Ibn Sīnā's take on the Porphyrian semantic theory comes close to a type-theoretic semantics like that in Frege.

Besides his lack of set-theoretic apparatus, and his over-reliance on linguistic theories,
he probably lacked sparring partners at his own level.
Also his elitist views about teaching probably prevented his Arabic successors from appreciating many of his insights.

So the aristotelian semantics with its inadequacies trundled along for another 900 years.