How did they manage before they had the notion of quantifier scope?

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How old is the notion of scope?

Benson Mates, *Stoic Logic* p. 30:

> [According to Sextus Empiricus (c. AD 200)] indefinite propositions lie in the scope of ... an indefinite particle; for instance, "Somebody is walking".

Sextus Empiricus says that the 'dominant' (*kurieúei*) part is the indefinite noun 'Somebody' — which probably just means it's the subject noun (cf. Apollonius Dyscolus). Hardly evidence that Sextus had the notion of scope!

But it is evidence of an empty patch in the history of logic.

Towards filling this patch, I will tell you what I know under five heads:

- I Frege's introduction of the notion of scope, 1879.
- II Extension to languages with phrase markers; Klima, Hodges, Reinhart.
- III Ibn Sīnā's problem with mixed quantifiers.
- IV Ibn Sīnā's dependency grammar preventing a solution.
- V Ibn Sīnā looks for a way out.

I. Frege introduces scope

Frege, *Begriffsschrift* (1879) §11:

> [The universal quantifier with a German letter variable] is necessary to delimit the scope [Gebiet] to which the generality indicated by the letter applies. Only inside its scope does the German letter have a fixed meaning [Bedeutung].

Probably the first occurrence in view of the confusions. The German letter doesn't have a meaning (or at least Frege never gave it one). Is the Gebiet a part of the symbolism or of the proposition expressed?

Frege, *Grundgesetze I* (1893) §8:

> We name what follows a [universal quantifier with a German letter] . . . the *scope* [Gebiet] of the German letter.

The definition is now purely syntactic, in terms of the shape of the formula.

Note the change: the quantifier itself is no longer in the scope of the variable.

Russell, *Mathematical logic as based on the theory of types* (1908):

> The *scope* of a real variable is the whole [propositional] function of which 'any value' is in question.

A real variable is a variable assumed to be quantified by an implicit universal quantifier on the whole formula; so its scope is by definition always the whole formula. Hardly a useful definition!

Hilbert and Ackermann *Grundzüge der Theoretischen Logik* (1928, largely reproducing Hilbert's lectures 1917–22)

> To each universal or existential quantifier occurring in a formula, there belongs a definite constituent part of the formula to which it relates.
> We shall call this part the *scope* [Wirkungsbereich] of the symbol in question.

The notion as defined in *Grundgesetze*, and by Hilbert and Ackermann, illustrates a type of linguistic definition:
A constituent *C*(ommander) of a sentence has some kind of control over a constituent *D*(omain).

Some syntactic examples:

- *C* is a noun and *D* is a clause which has *C* as its subject noun; *C* governs the inflection of any verbs within *D*.
- *C* is a negation and *D* is the part of the sentence where, because of *C*, we can have 'negative polarity' words like 'any'.
- *C* is a noun phrase and *D* is the part of the sentence within which every pronoun co-referential with *C* has to be reflexive.

There is a widespread consensus (which you are welcome to disagree with) that we should try to give *structural* characterisations of what commands what.

The structure can be syntactic, but for some cases (e.g. coreference, logical scopes) one may have to look at the underlying structure of meanings.

That's *Begriffsschrift* for Frege, or LF in Chomsky circles. Ibn Sīnā had a corresponding notion of 'composition of meanings' — more on this below.

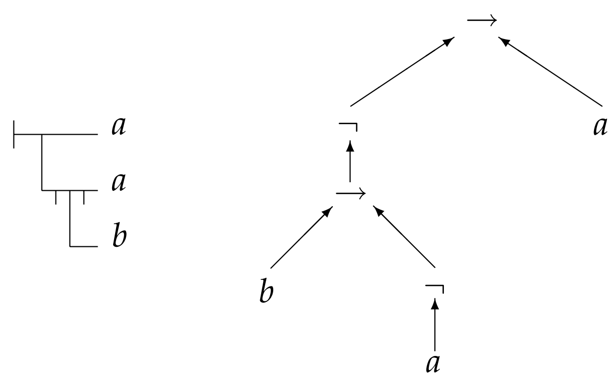
The language of Frege's *Begriffsschrift* has a dependency grammar.
Each of its sentences is its own grammatical analysis.

In a dependency grammar we are given a binary asymmetric relation 'depends on' between words, and a family of *dependency rules* of the form 'The words $a_1, \ldots, a_n$, in this order, depend on b'.
A rule may also give b a position among the $a_1, \ldots, a_n$.

A *stemma* (for a dependency grammar) is a tree diagram where the nodes are words, the edges (running upwards) express 'depends on', there is a unique top node, and the nodes immediately below any node are given by one of the dependency rules.

For each grammatical sentence there is a stemma; the sentence is the nodes of the stemma, written in the left-right order given by the stemma.
A *constituent* of the sentence is the segment got by taking a node and the words below or at that node.

Dependency stemma for $(\neg(b \to \neg a) \to a)$ (Frege's version on left):



Here the constituents are the subformulas.

In *Grundgesetze*, the scope of a variable in a formula is the subformula determined by the node immediately below the node of the quantifier introducing the variable.

This needs some clumsy clarification if there is more than one quantifier with the same variable.
Hilbert and Ackermann repair this fault by talking of the scope of the *quantifier occurrence*.

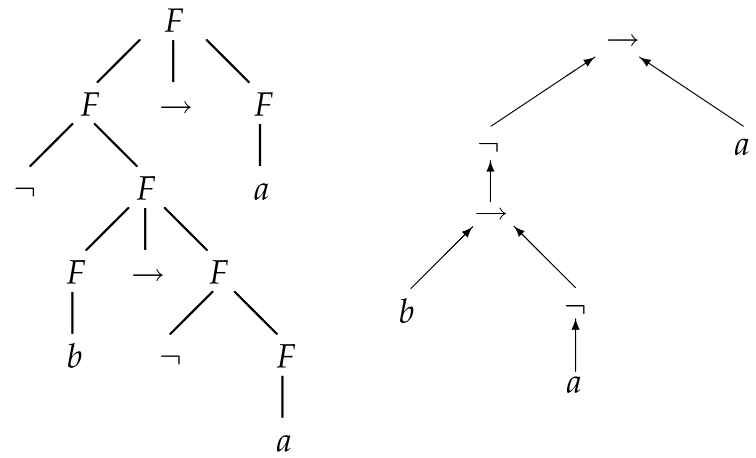
The language of Hilbert and Ackermann is also given by a dependency grammar.
But they flatten it down, putting the quantifier to the left of the subformula below it in the grammar.
So the scope of the quantifier is the subformula immediately to the right of it.

II. Extension to phrase markers

Dependency grammars are not much favoured today.
They can't express that an element (e.g. a quantifier, a modality) has a whole constituent dependent on it, as opposed to just the 'head' of the constituent.

Instead linguists now tend to use Chomsky's *phrase markers*,
which are trees with the words at the bottom nodes,
and the higher nodes correspond 1–1 to the constituents.

Phrase-marker of $(\neg(b \to \neg a) \to a)$ (with stemma to compare):



Question: What is the appropriate notion of the scope of an element in a phrase marker?
NB The only constituent 'below' a word is the word itself.

The question was almost answered by Edward Klima, 'Negation in English', in Fodor and Katz, *The Structure of Language* (1964), p. 297. He wanted a notion of 'constituent X is in the scope of negation Y'. With the terminology slightly adjusted, his definition reads:

> A node is *in construction with* another node if the former is below the first branching node that is above the latter.

I don't know if Klima meant 'another' literally. Assuming not, the words that are in construction with a node *N* are precisely those in the smallest constituent that contains both *N* and a word not in *N*.

My *Logic* (1976) used phrase markers throughout. On p. 58 I defined:

> [If] P is a part of [the sentence] S consisting of one or more words, but not the whole of S, then we define the *scope* of P to be the smallest constituent of S which contains both P and something else besides.

I had read Klima but forgotten his definition.

Tanya Reinhart, *Anaphora and Semantic Interpretation* (1983) p. 18f, gave a definition which has been hugely influential in linguistics:

> Node A *c-commands* node B iff the branching node most immediately dominating A also dominates B. . . . The domain of a node A consists of all and only the nodes c-commanded by A.

So A c-commands B if and only if B is in construction with A.

('c' is for 'constituent'.)

Remarks:

1. Reinhart had also read Klima. She remarks that Klima added a further condition making the nodes incomparable; but from my copy of Klima it seems that the further condition was added by Reinhart herself in her 1976 PhD thesis, and dropped in her book.

2. Reinhart also remarks that her definition of domain 'makes no mention of linear ordering'. This will be important for us.

3. By my definition a quantifier is within its own scope. This seems to agree with Frege *Begriffsschrift* but not *Grundgesetze*.

Robert May, *Logical Form, Its Structure and Derivation* (1985) p. 5 proposes using Reinhart's *domain* in LF as a definition of *(logical) scope*.

The difference from my definition is that for May the scope (e.g. of a quantifier) is a set of nodes or constituents, whereas for me it's a single constituent.

In practice it seems to me May operates more with my definition than with his own. But there are probably linguistic subtleties that I miss.

Ibn Sīnā, Persia 980–1037

III. Ibn Sīnā's logical problem

Ibn Sīnā was interested in how scientists talk and think. He noticed that sentences of the form 'Every *A* does *X*' may need different quantifier analyses.

Take for example the implied temporal quantifiers in:

- Every horse is an animal.
- Every animal breathes.
- Everybody who writes moves his hand.
- Everything that breathes in breathes out.

An important part of understanding a sentence is knowing how to negate it.

Arabic has a sentence negation operator:
put *laysa* in front of the sentence.
But Ibn Sīnā is never happy with this form. He tries to move the negation inwards, using De Morgan-like rules.

In examples above, this involves moving a negation inwards where the sentence has a $\forall$ quantifier and a $\exists$ quantifier.

Question One: What happens to a quantifier when it is moved out of the scope of a negation?

Several of the sentences that he considers have a universal subject quantifier and an existential time quantifier.

Ibn Sīnā knows that in such cases we can express the existential quantifier through an added condition. E.g.

- Every moon gets eclipsed.
- For every moon x there is a time t such that x is eclipsed at t.
- For every moon x and every time t, if the earth is between the sun and x at time t, then x is eclipsed at time t.

Question Two: How do these analyses relate to the scopes of the two quantifiers?

Ibn Sīnā never shows any awareness of either of these questions.
In fact he never shows any awareness of scope in any form.

So we ask: (IV) what is getting in the way of his having a notion of scope? and (V) how does he manage to survive without a notion of scope?

IV. The obstacle to a notion of scope

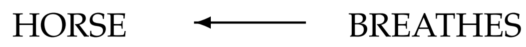
Ibn Sīnā was bilingual Arabic-Persian.
Arabic puts the verb at the beginning of a sentence, Persian at the end.

Ibn Sīnā deduces:

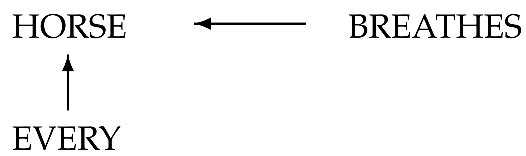
> In a sentence the subject and the predicate don't have to come in a particular order by nature. (Ibn Sīnā ^{c}*Ibāra* 31.4.)
> Speech doesn't have to have an established natural ordering. (Ibn Sīnā *Maqūlāt* 130.1)

But Ibn Sīnā also believes that the meaning of a sentence is structured, *but as a tree, not linearly ordered*.
It is built up from the meanings of the words in the sentence.
In effect, Ibn Sīnā uses a dependency grammar of meanings.

The core construction is between a subject term and a predicate term:



Quantifiers attach to the subject term:



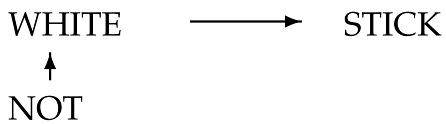
Question: To negate this, where do we attach NOT so as to make it apply to the whole formula?

Recall Frege's answer:
We don't attach NOT to anything.
We attach HORSE to NOT (so NOT becomes top node).

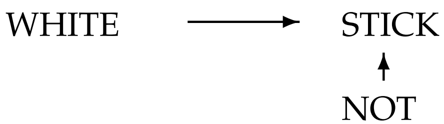
This idea never crossed Ibn Sīnā's mind.
To him and linguists of his time, it was obvious that NOT is attached somewhere in the proposition.

A simpler example (which already defeats Ibn Sīnā):

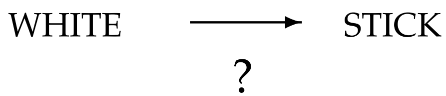
Not-white stick



White non-stick



Not a white stick



Now when we introduce time to sentences:

how do we attach the quantifier 'SOME t' to signal the difference between

Every horse breathes at some time.

and

There is a time when every horse breathes. ?

V. Ibn Sīnā looks for a way out

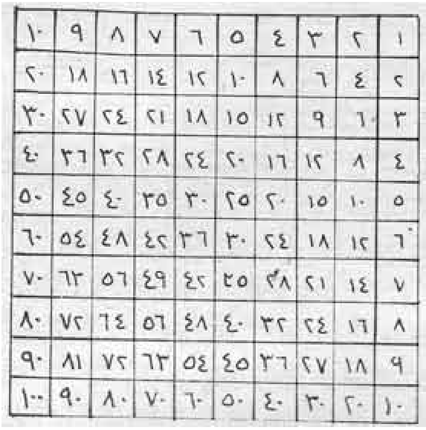
Ibn Sīnā knew he was in a hard place:

> It would be appropriate for us to speak warily ...
> This makes difficulties for us ...
> We land ourselves in the following difficulties when we introduce this point ...

Unlike Frege, he never diagnosed the inadequacy of his grammar of concepts.
Instead he did the following.

He considered the horses and the moments of time as a 2-dimensional array.

He had no problems thinking this way.
Diagram from his précis of Nicomachus:



١٠	٩	٨	٧	٦	٥	٤	٣	٢	١
٢٠	١٨	١٦	١٤	١٢	١٠	٨	٦	٤	٢
٣٠	٢٧	٢٤	٢١	١٨	١٥	١٢	٩	٦	٣
٤٠	٣٦	٣٢	٢٨	٢٤	٢٠	١٦	١٢	٨	٤
٥٠	٤٥	٤٠	٣٥	٣٠	٢٥	٢٠	١٥	١٠	٥
٦٠	٥٤	٤٨	٤٢	٣٦	٣٠	٢٤	١٨	١٢	٦
٧٠	٦٣	٥٦	٤٩	٤٢	٣٥	٢٨	٢١	١٤	٧
٨٠	٧٢	٦٤	٥٦	٤٨	٤٠	٣٢	٢٤	١٦	٨
٩٠	٨١	٧٢	٦٣	٥٤	٤٥	٣٦	٢٧	١٨	٩
١٠٠	٩٠	٨٠	٧٠	٦٠	٥٠	٤٠	٣٠	٢٠	١٠

For some sentences we quantify over all points of the array. E.g.

Everybody who writes moves his hand.

But for ‘Every horse breathes’ we can’t do this. ‘The quantifier is on HORSE, not on the two things together.’

So instead he quantifies over a *thinned-out array*, where for every horse there are times, but in general different times for different horses.

Success!
Ibn Sīnā can give a truth condition for ‘Every horse breathes’,
and he can do it without putting any ordering on the quantifiers at concept level.

Modern logicians who know Henkin’s Skolem function semantics for partially ordered quantifiers will see the resemblance.
The array plays the role of the Skolem function.
Again note the importance of *not having a linear order*.

Next question: how to negate the sentence.
Ibn Sīnā has no concept of negating a 2-dimensional array.
So he tries to give a truth condition for the negation, using the same sort of array.
He fails, and he can’t see why.

At this point he falls back on commonsense reasoning.
He uses this as an example of the dangers of artificial languages in logic.
All our commonsense intuitions are about our own language;
we have no reliable intuitions about artificial ones.

There is a provisional translation of Ibn Sīnā's discussion on the web at

`http://wilfridhodges.co.uk/arabic11.pdf`

Warning: It's tough going and much of it makes little sense yet.
In this lecture I tried to trace back from it to things in Ibn Sīnā that we do understand.

For completeness: Walter Burley in his longer *De Puritate Artis Logicae* p. 27 (early 14th c.) remarks that 'a negation doesn't have *dominium* over what precedes it'.
He seems to mean scope in some sense, but I couldn't find any other use of this word *dominium* in Burley.
Maybe some medievalist can trace it back further.

In any case Burley's claim is quite false for his own native language of Middle English.
There is a little further discussion in my 'Detecting the logical content: Burley's "Purity of Logic" ', *We Will Show Them! Essays in Honour of Dov Gabbay II*, ed. S. Artemov et al., College Publications 2005, pp. 70f.

A remark of Padoa:

> 'You don't understand a definition until you know how to do without it.'

He meant that definitions are eliminable.
But maybe his remark has a deeper meaning.

For example:

> To understand scope, try doing without it.