The combinatorics of imperfect information

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February 2000

Leon Henkin (1961) introduced quantifiers as follows:

$$\left.\begin{matrix} (\forall x) & (\exists v) \\ (\forall y) & (\exists w) \end{matrix}\right\} \phi(x, y; v, w).$$

The intended meaning is:

There are functions f, g such that

$$(\forall x)(\forall y)\ \phi(x, y; f(x), g(y)).$$

Intuitively: For all x and y there are v depending only on x and w depending only on y such that $\phi(x, y; v, w)$.

Example (Ehrenfeucht 1958):

$$\left.\begin{matrix} (\forall x) & (\exists v) \\ (\forall y) & (\exists w) \end{matrix}\right\} [((x = y) \leftrightarrow (v = w)) \wedge \phi]$$

says

$$(\exists f)(\exists g)[(\forall x)(\forall y)((x = y) \leftrightarrow (f(x) = g(y))$$

$$\wedge (\forall x)(\forall y)\phi(x, y; f(x), g(y))]$$

i.e.

$$(\exists \text{ injective } f)(\forall x)(\forall y)\phi(x, y; f(x), f(y))$$

But

$$(\exists z)(\exists \text{ injective } f)(\forall x)((Rx \to Rf(x))$$

$$\wedge f(x) \neq z \wedge Rz)$$

holds if and only if R is infinite.

FACT (Walkoe and Enderton independently, 1970)

There is an algorithm for translating each sentence

$$(\exists f_1)\ldots(\exists f_m)\phi$$

with ϕ first-order, to a sentence

$$\left.\begin{matrix} (\forall x_{11}) & \ldots & (\forall x_{1n}) & (\exists v_1) \\ & \ldots & & \ldots \\ (\forall x_{m1}) & \ldots & (\forall x_{mn}) & (\exists v_m) \end{matrix}\right\} \psi$$

with ψ first-order, which has exactly the same models.

Blass and Gurevich (Henkin quantifiers and complete problems, *Annals of Pure and Applied Logic* 32 (1986) 1–16) study Henkin sentences on finite structures, allowing existential quantifiers over truth-values instead of elements.

FACT (Blass and Gurevich)

'Almost any [branching] quantifier, applied to quantifier-free first-order formulas, suffices to express an $\mathcal{NP}$-complete predicate.
The remaining non-linear quantifiers express exactly co-$\mathcal{NL}$ predicates.'

Jaakko Hintikka 1973 gives a game semantics for first-order logic, using players $\forall$, $\exists$.

Given a structure A:

$\forall x$ means player $\forall$ chooses an element a of A for x.
$\exists x$ means player $\exists$ chooses an element a of A for x.

$(\phi_0 \wedge \phi_1)$ means player $\forall$ chooses one of $0, 1$.
$(\phi_0 \vee \phi_1)$ means player $\exists$ chooses one of $0, 1$.

$\neg\phi$ means the players change places.

$P(x_0, \ldots, x_{n-1})$ is a win for $\exists$ if $(a_0, \ldots, a_{n-1})$ is in P^A, and a win for $\forall$ otherwise.

THEOREM. A sentence ϕ is (Tarski-)true in a structure A if and only if player $\exists$ has a winning strategy for the game $G(\phi, A)$ just described.

Remarks:

1. The clause for $\neg$ works only because the game is determined (by the Gale-Stewart theorem, since it has finite length).

2. Unlike the traditional Tarski semantics, the game semantics gives no direct interpretation for formulas that are not sentences.
(It is not 'compositional'—though this term has led to various confusions.)

In about 1990, Hintikka suggested how to adapt game semantics to Henkin's logic.

'y is independent of x' is taken to mean: a strategy for choosing an element for y must not depend on the element chosen for x.
(So games are of imperfect information.)

Thus for Henkin's sentence, Hintikka writes

$$(\forall x)(\exists v)(\forall y)(\exists z/\forall x)\phi.$$

A strategy for $\exists$ is exactly a pair of functions $f(x), g(y)$;
it is winning if and only if the Henkin condition on Skolem functions holds.

Example 1

$$(\forall x)(\exists y/\forall x)x \neq y$$

Neither player has a winning strategy for this sentence in a structure of more than one element.
Hence Hintikka's semantics can't express negation.
In particular there is no game that a player wins if and only if the Henkin sentence is *false*.

So Hintikka has to restrict where negation can occur.

Example 2

$$(\forall x)(x = 0 \vee (\exists y/\forall x) y \neq x).$$

Winning strategy for $\exists$:

At $\vee$, if 0 was chosen for x, choose '$x = 0$'; otherwise choose '$(\exists y/\forall x) y \neq x$'.

At $(\exists y/\forall x)$, choose 0.

NOTE: $\exists$ uses her choice at $\vee$ to send a signal to her choice for y.
(Hodges:) Therefore we should allow $\exists$ to use her own earlier choices to signal to later ones.

Example 3

$$(\forall x)(\exists y)(\exists z/x)\ x = z.$$

As before, $\exists$ has a winning strategy by signalling to herself.

In this syntax, strategies for a player can use choices of either player at earlier variables x, except where these earlier variables x are slashed.

Two proposals of Abramsky

1. 'If information is hidden at some stage, it should remain hidden.'

This blocks the example above;
but in fact it reduces Hintikka's language to ordinary first-order.

2. 'The interpretation of the Henkin quantifier should be symmetrical for $\forall$ as well as $\exists$, i.e. $\forall$ should be able to choose which branch to take first. This would add a multiplicative feature to the logic.'

This sentence does it:

$$((\forall x)(\exists v)(\forall y/v)(\exists w/xv)\phi$$

$$\wedge$$

$$(\forall y)(\exists w)(\forall x/w)(\exists v/yw)\phi)$$

In 1996 I gave a semantics for this extension of Hintikka's logic, that interprets formulas by induction on their complexity, like the usual Tarski semantics.

Purposes:

- To understand the logic.
- To see how to restore classical negation.
- To refute Hintikka's repeated claim that no such semantics can be found.

In 1997, urged by Sandu, I adapted this semantics to Hintikka's syntax. (Not trivial but not hard.)

Given a formula $\phi(x, y, \ldots)$, we don't (as in Tarski) ask what assignments of elements to variables satisfy the formula.

Instead we ask what *nonempty sets of assignments* satisfy it uniformly (i.e. in a way that doesn't depend on hidden information). Call such a set a *trump*.
A *cotrump* is a set with the dual property (corresponding to swapping $\forall$ and $\exists$).

So the semantics for ϕ consists of an ordered pair of disjoint sets of nonempty sets of assignments, both closed under subsets.
Over a finite structure, we can show that every such ordered pair is the interpretation of some formula.

In the case of $\phi(x)$ with just x free, the number of such pairs in a structure of n elements is as follows:

n	number of pairs
1	3
2	11
3	55
4	489
5	17,279
6	15,758,603
7	4,829,474, 397,415
8	112,260,874, 496,010,913, 723,317

(Joint with Peter Cameron, using a 200-hour calculation on a Cray)

A general theorem shows this is the *smallest* semantics that will make enough distinctions between formulas.

For Hintikka's logic we need only trumps, and the corresponding figures tend to half those above as n tends to ∞.

The formulas which need this many possible interpretations are all (essentially) of the original Henkin form,
but further sentences are needed to show that no simpler semantics will work.

Related work in progress:

Caicedo and Krynicki give a prenex normal form for Hintikka's logic.

Väänänen gives a game-theoretic semantics for Hintikka logic using games of perfect information, but the players choose sets of assignments rather than single assignments.

Abramsky sketches a proof-theoretic semantics for a variant of Hintikka's logic.