Four paradigms for logical games

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Dates and references

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Logical games

A logical game is a pair (G, τ) where

- G is a nonempty set of sequences of length $\leqslant \omega$, which is closed under initial segment and limit;
- $\tau : G \to \{1, 2\}$.

Then G forms a tree branching upwards, under the partial ordering

$$\bar{a} \preccurlyeq \bar{b} \;\Leftrightarrow\; \bar{a} \text{ is an initial segment of } \bar{b}.$$

Maximal elements of G are *plays* of G; the remaining elements of G are *positions* of G. The numbers 1, 2 are called *players*. A play $\bar{a}$ is a *win for* $\tau(\bar{a})$. A position $\bar{a}$ is a *turn of* $\tau(\bar{a})$. Logical games count as zero-sum: the *payoff* is a win for one player and a lose for the other.

A *strategy* for player π is a function

$$\Sigma^\pi : \text{(The set of turns of } \pi) \;\to\; G$$

such that for every turn $\bar{a}$ of π, $\Sigma^\pi(\bar{a})$ is immediately above $\bar{a}$ in G. A position or play $\bar{a}$ *follows* the strategy Σ^π if for every turn $\bar{a}\restriction n$ of π with $n < \ell h(\bar{a})$, $\Sigma^\pi(\bar{a}\restriction n) \preccurlyeq \bar{a}$. The strategy Σ^π is *winning* if every play that follows Σ^π is a win for π. The game (G, τ) is *determined* if there exists a winning strategy for one of the players.

We topologise the set of plays by taking as basic open sets the sets of the form

$$\{\bar{a} \in \text{plays } : \bar{b} \preccurlyeq \bar{a}\} \quad (\bar{b} \text{ a position}).$$

The *Gale-Stewart Theorem* (1953) says that if for some player π the set of wins for π is open, then the game is determined. Hence if all plays are finite, the game is determined.

Suppose Σ^1 and Σ^2 are strategies for players 1, 2. Then there is a unique play that follows both strategies. (Hence at least one of the strategies is not winning!) Von Neumann and Morgenstern give for each game a *strategic* (they say *normalized*) form. This form is the function which, to each pair Σ^1, Σ^2, assigns the payoff of the unique play that follows both strategies. Compared with the strategic form, the original game is said to be in *extensive* form. At least until recently, logicians have always used the extensive form.

Strategies can be simplified in three ways.

(1) If the position $\bar{a}$ is a turn for π, then $\Sigma^\pi(\bar{a})$ has the form

$$\Sigma^\pi(\bar{a}) = \bar{a}^\frown c$$

for some c. Instead of defining Σ^π, we can define

$$\sigma^\pi(\bar{a}) = c.$$

Then we can translate the notions of a position or play *following* a strategy, and of a *winning* strategy, from Σ^π to σ^π.

(2) For each strategy σ^π define the *core* σ_0^π to be the restriction of σ^π to positions which follow σ^π. The question whether σ^π is winning is determined by its core.

(3) Given a player π and a position $\bar{a} = (a_0, \dots, a_{\ell-1})$, define $\bar{a}^{(\pi)}$ to be the sequence $(b_0, \dots, b_{\ell-1})$ where for each $n < \ell$,

$$b_n = \begin{cases} \star & \text{if } \bar{a}\restriction n \text{ is } \pi\text{'s turn,} \\ a_n & \text{otherwise.} \end{cases}$$

Then a core σ_0^π is recoverable from the function $\sigma_0^{(\pi)}$ defined on the set of sequences $\bar{a}^{(\pi)}$ (where $\bar{a}$ is a position that follows σ^π and is π's turn) by:

$$\sigma_0^{(\pi)}(\bar{a}^{(\pi)}) = \sigma^\pi(\bar{a}).$$

(A strategy can be defined as a function of moves of the other player.)

Von Neumann and Morgenstern also considered games of *imperfect information*, where the domain of a strategy of π is a surjective image of the set of turns of π. In this case simplification (3) fails in general, because information suppressed by putting $\star$ may be needed to define the strategy (the phenomenon of *signalling*).