The geometry of music

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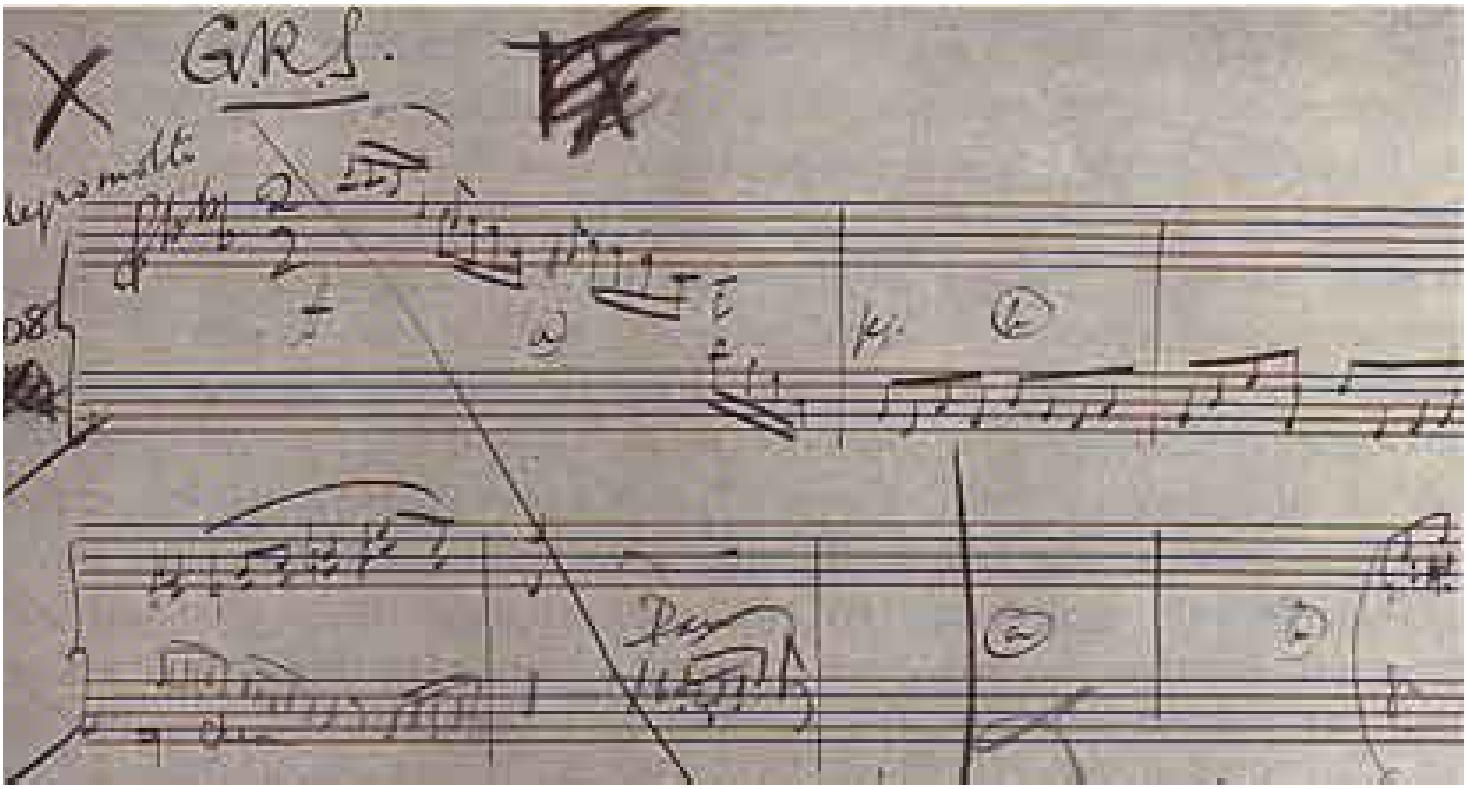
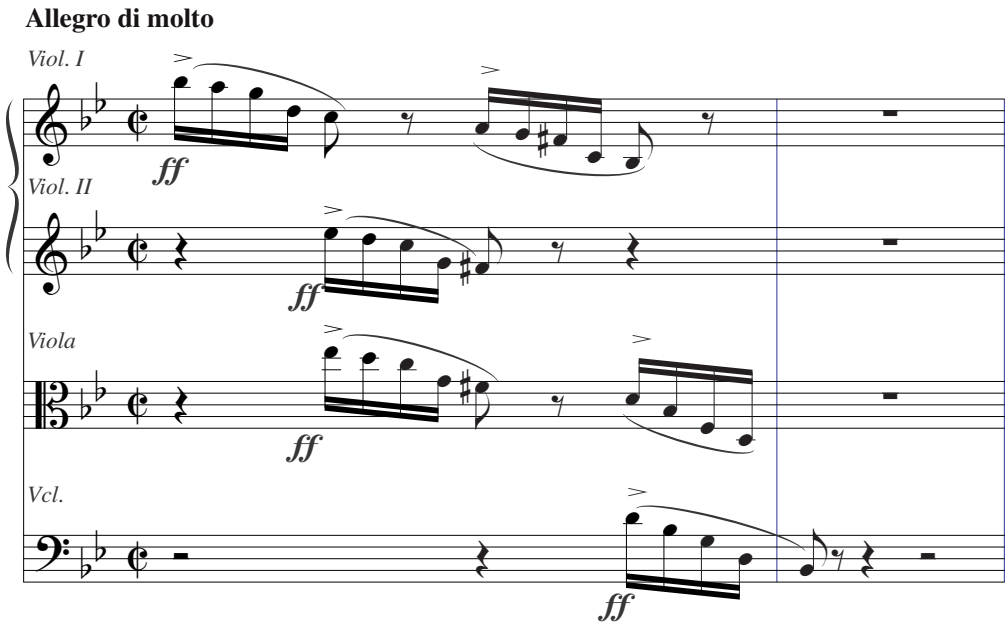
Dartmoor, November 2009

http://wilfridhodges.co.uk



Edward Elgar, ‘Enigma Variations’



Music takes place in pitch-time space.

Time goes → on the page, pitch usually goes ↑.

We are limited to a small region of pitch,
say 25Hz to 5000Hz;
and usually a small region of time too.

Even in the bounded pitch region it sounds
strange if we go outside a limited set of pitches.

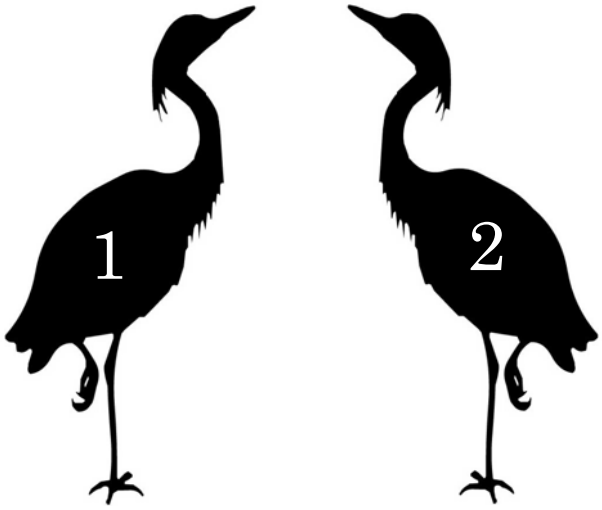
Felix Klein 1872

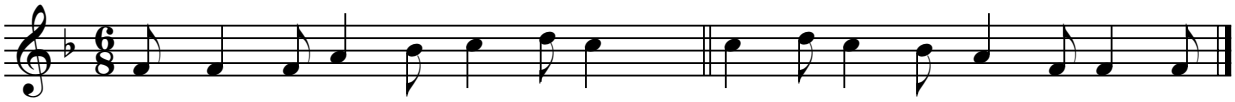
Geometry studies a space through its transformations. The contents of the space are classified by how they transform.

The multiplication table of Klein's Four-group:

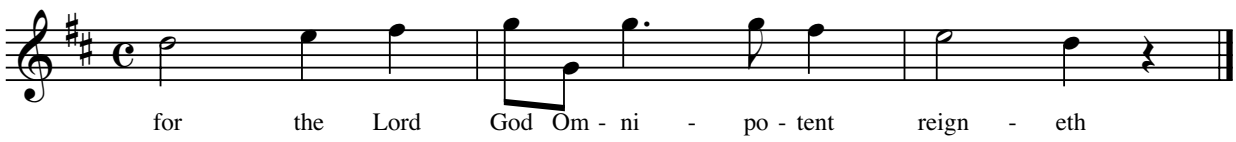
$\times$	I	M_h	M_v	R
I	I	M_h	M_v	R
M_h	M_h	I	R	M_v
M_v	M_v	R	I	M_h
R	R	M_v	M_h	I

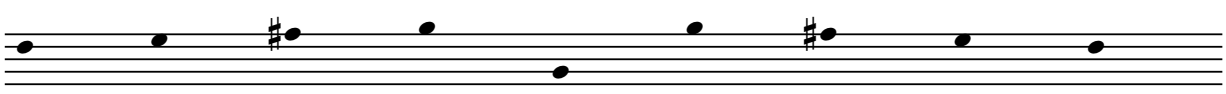
Horizontal mirror M_h



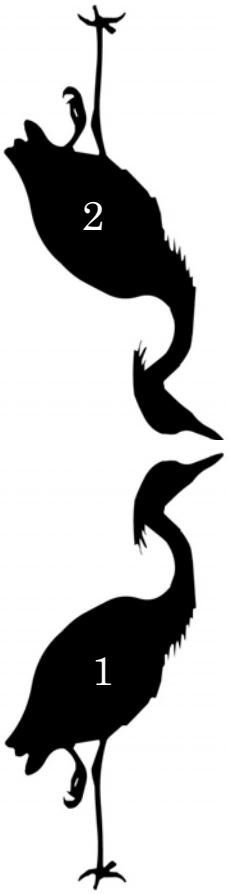


Georg Handel, ‘Hallelujah Chorus’

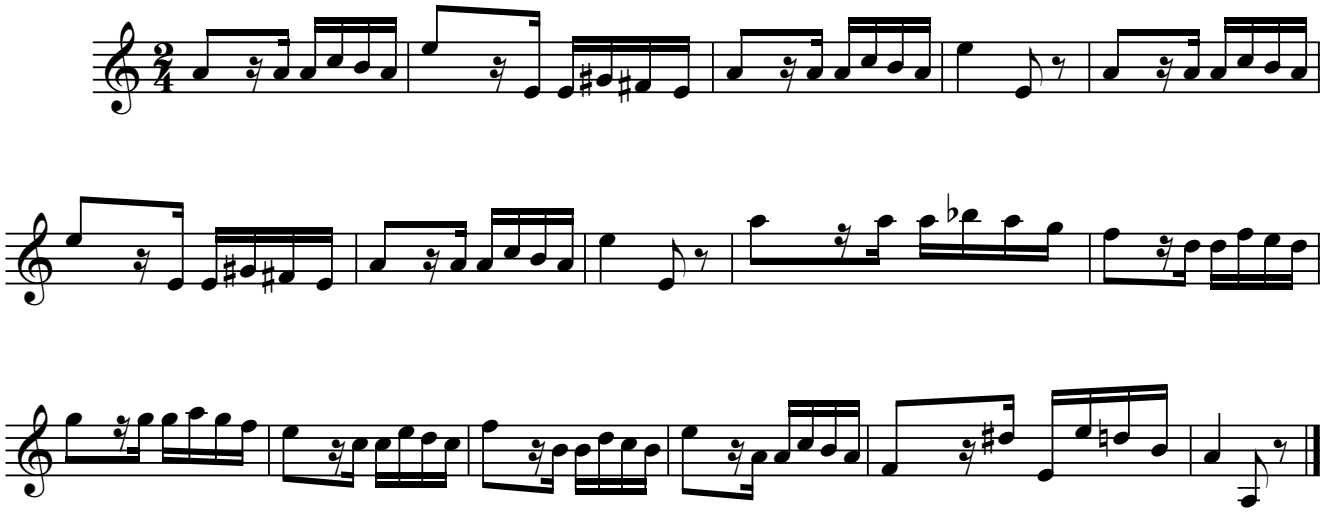




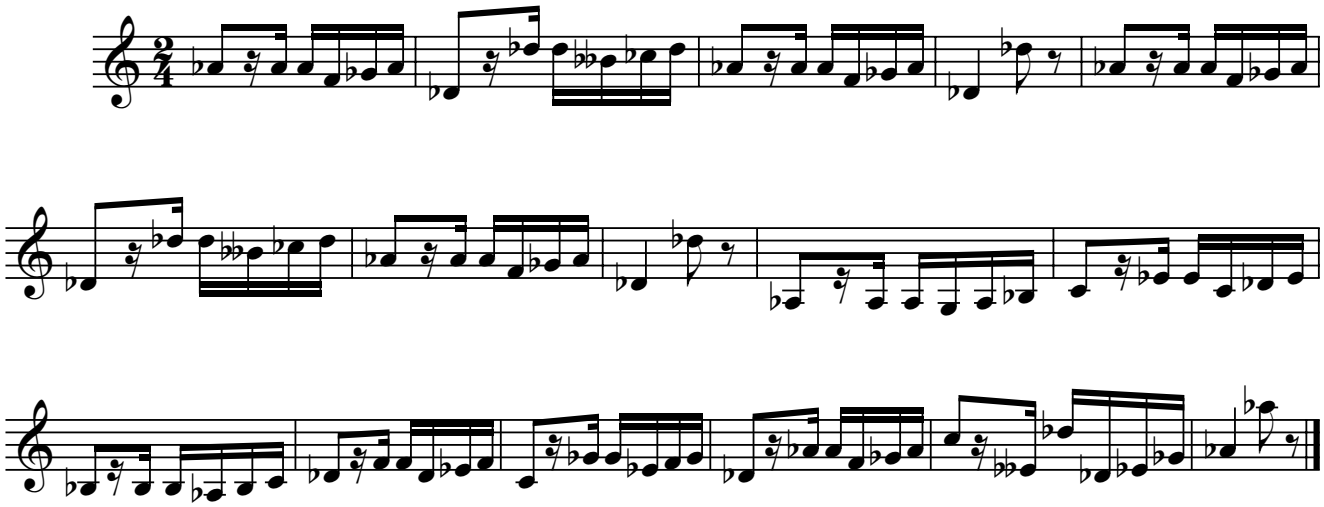
Vertical mirror M_v



Paganini, ‘Capriccio 24 for Violin’



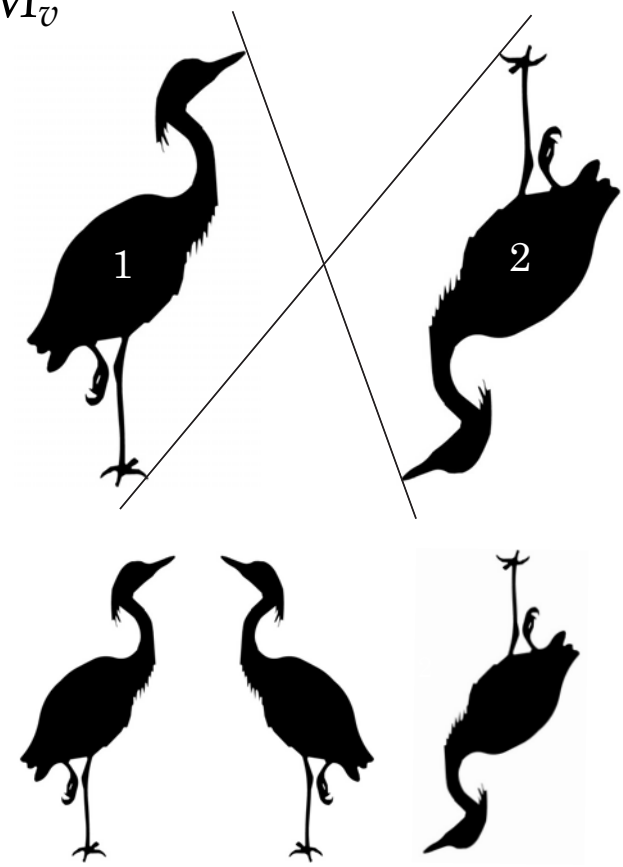
Sergei Rachmaninov,
‘Variations on a Theme of Paganini’



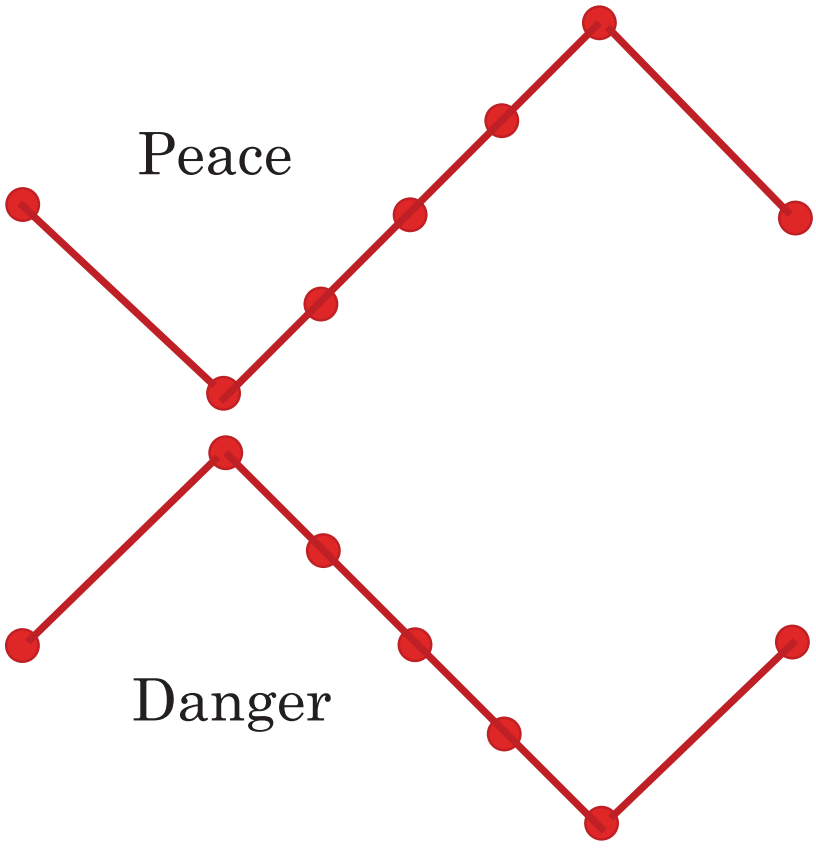
The Classic FM Hall of Fame in 2008
put the upside down version 15th.

The right way up version was nowhere.

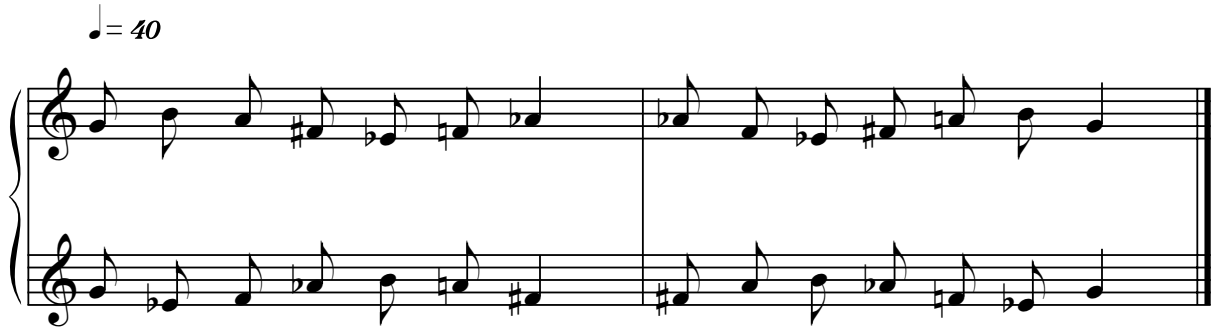
Rotation $R = M_h \times M_v$



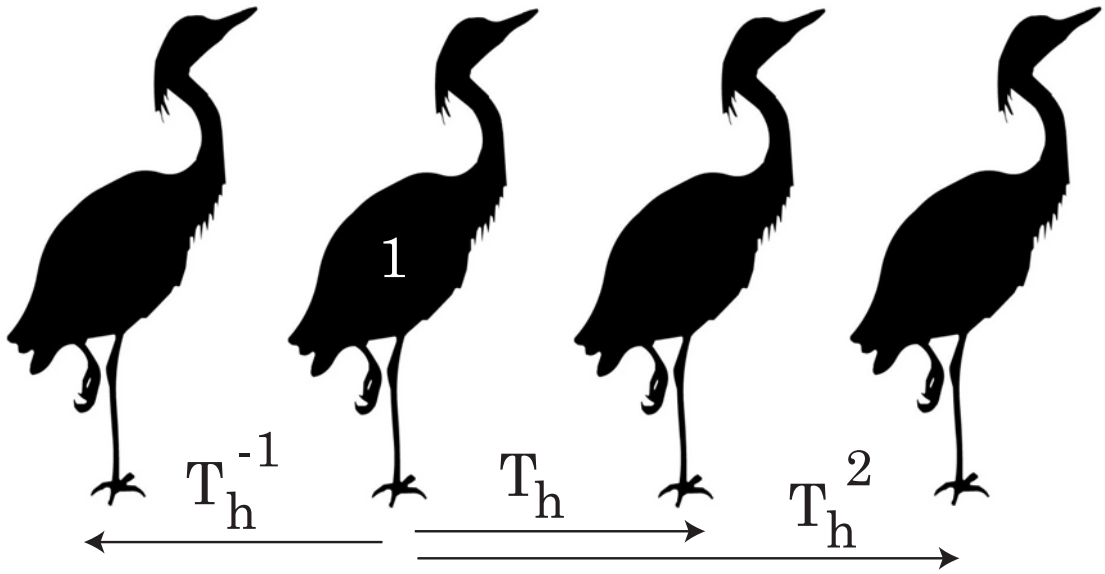
Nikolai Rimsky-Korsakov,
'The Golden Cockerel'



John Tavener, ‘The Lamb’



T_h moves everything to the right
(through some fixed distance)



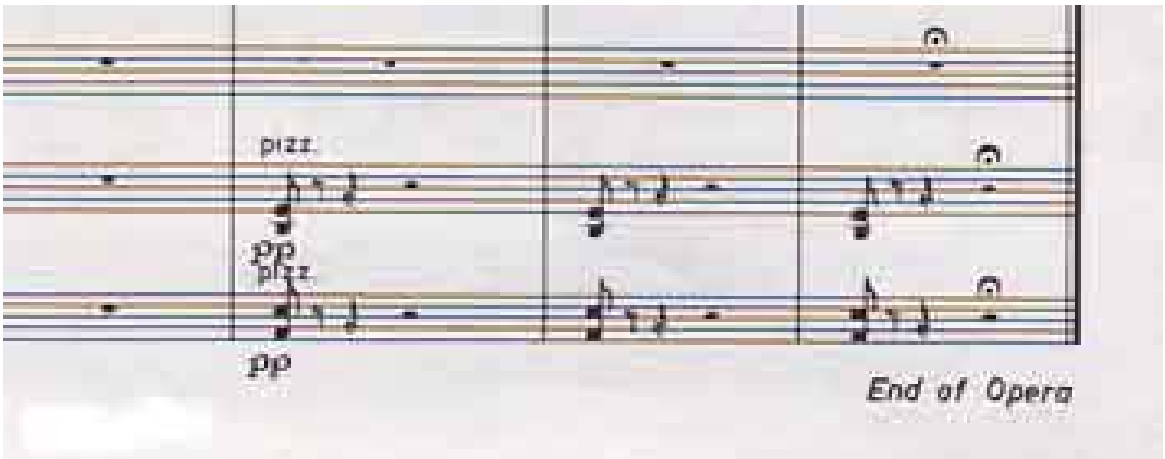
Likewise T_v moves everything vertically.

Doing T_h once is called a 'repeat'.

Doing both T_h and T_h^2 gives three copies in a row, suggesting more, so

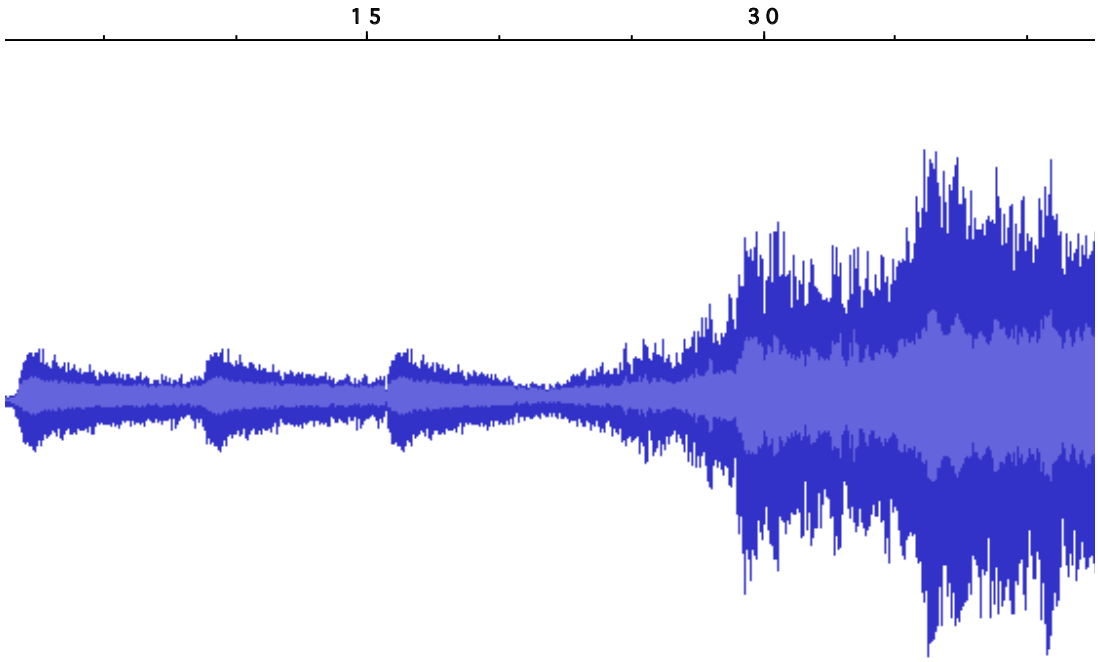
- infinite time
- constancy
- death
- excitement
- boredom
- ??

The musical dot-dot-dot:
Benjamin Britten, ‘Peter Grimes’



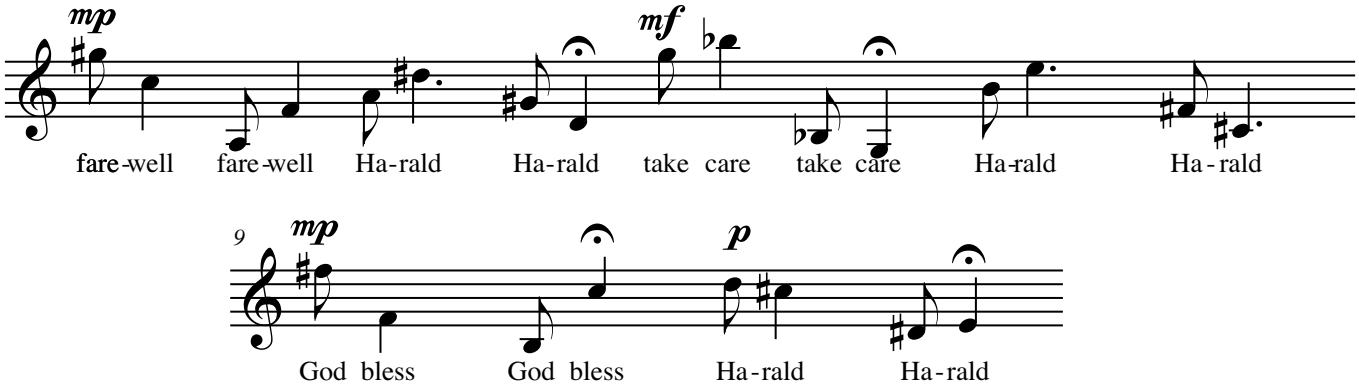
‘In ceaseless motion comes and goes the tide’

Pärt, ‘Cantus in Memoriam Benjamin Britten’

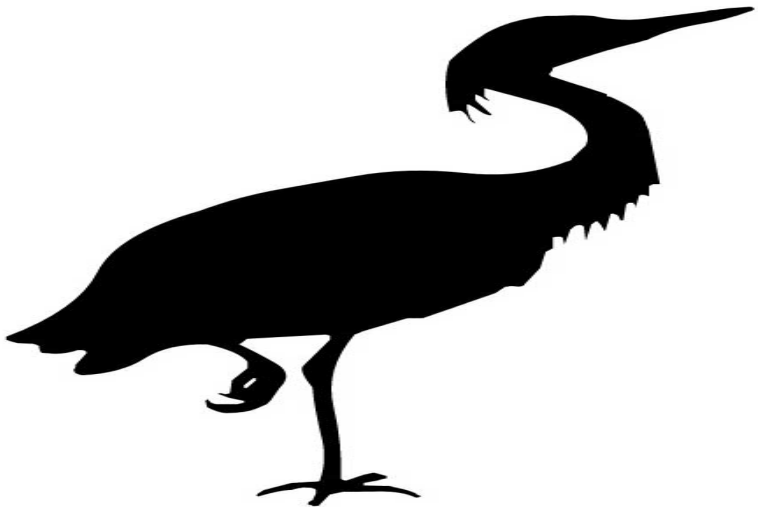


$T_h \times M_v$ (glide reflection)

Judith Weir, ‘King Harald’s Saga’



Horizontal dilation D_h (in some fixed ratio)



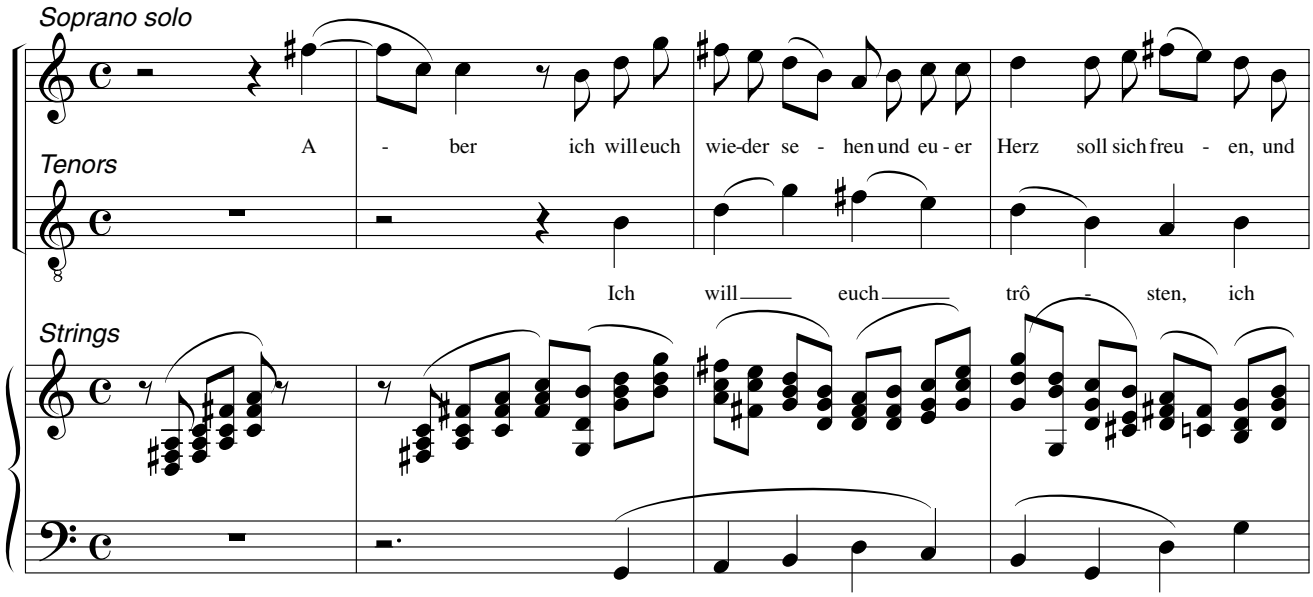
$T_h \times T_v \times D_h$

has a second voice repeating the theme later,
at a different pitch and a different speed.

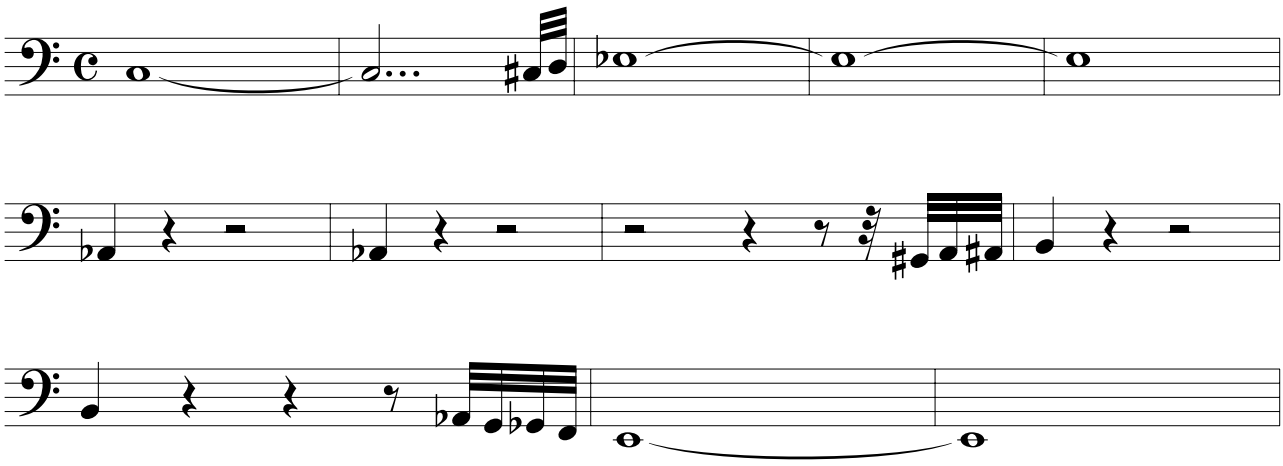
This device is called *mensural canon*.

It was popular in the late Middle Ages.
We consider three modern examples,
hugely different from each other.

(1) Johannes Brahms, ‘A German Requiem’
Two voices

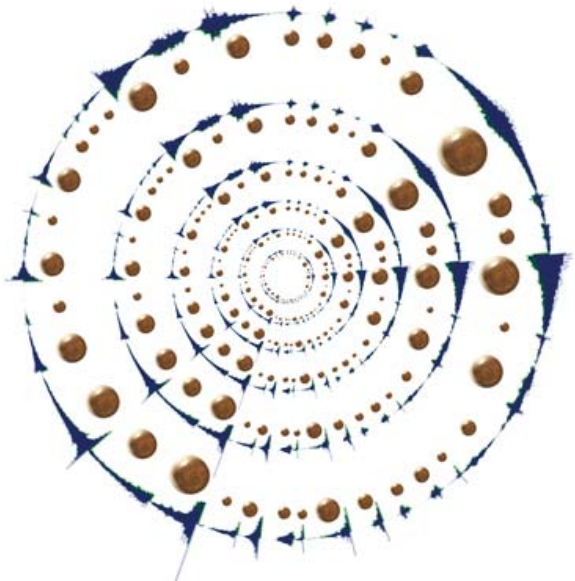


(2) Conlon Nancarrow,
'Study for Player Piano 36'
Four voices



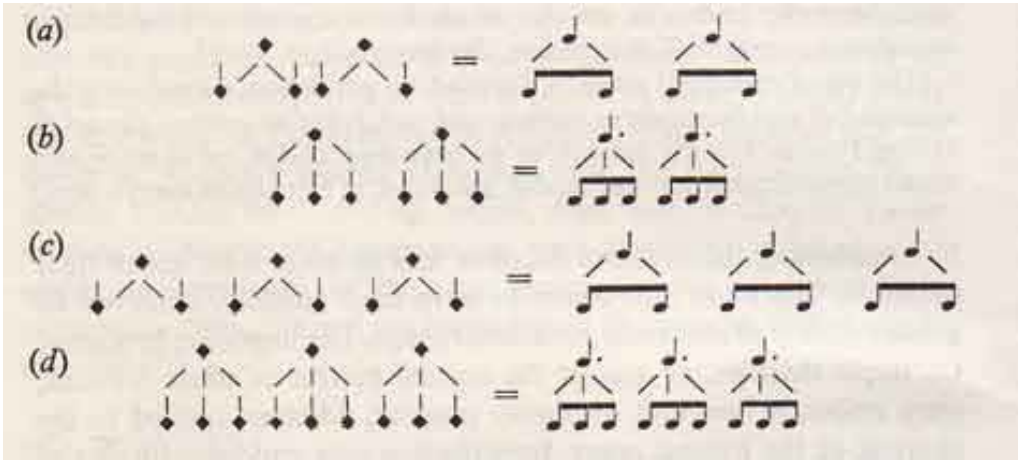
(3) Jem Finer 'Longplayer'

A piece that started on 1 January 2000 and takes exactly 1000 years to perform. There are 6 voices, all but one repeating many times.



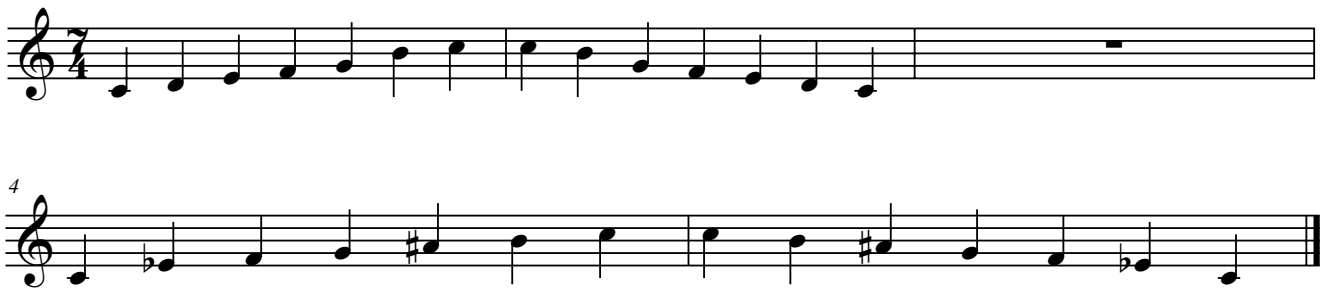
Musicians take a *framework* of possible pitches, rhythms etc., and put music into it.
Geometry applies at both levels.

Before the 20th century, the framework was determined by custom. Philip de Vitry (14th c):



Today composers often choose their framework.

Two different scales that contain exactly the same intervals the same number of times.



(Joint work in progress with Patrick Ozzard-Low.)

John Fauvel, Raymond Flood and Robin Wilson eds., *Music and Mathematics: From Pythagoras to Fractals*, Oxford University Press 2003

Robin Wilson, who has taught us all that bringing mathematics and music together helps us to appreciate both of them better